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TOPICAL REVIEW

Nonequilibrium statistical mechanics of the zero-range process and related models

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Online at stacks.iop.org/JPhysA/38/R195**Abstract**

We review recent progress on the zero-range process, a model of interacting particles which hop between the sites of a lattice with rates that depend on the occupancy of the departure site. We discuss several applications which have stimulated interest in the model such as shaken granular gases and network dynamics; we also discuss how the model may be used as a coarse-grained description of driven phase-separating systems. A useful property of the zero-range process is that the steady state has a factorized form. We show how this form enables one to analyse in detail condensation transitions, wherein a finite fraction of particles accumulate at a single site. We review condensation transitions in homogeneous and heterogeneous systems and also summarize recent progress in understanding the dynamics of condensation. We then turn to several generalizations which also, under certain specified conditions, share the property of a factorized steady state. These include several species of particles; hop rates which depend on both the departure and the destination sites; continuous masses; parallel discrete-time updating; non-conservation of particles and sites.

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(Some figures in this article are in colour only in the electronic version)

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1. Introduction

The statistical mechanics of nonequilibrium systems is required to understand the macroscopic behaviour of processes occurring throughout physics, chemistry, biology, and even sociology and economics. Nonequilibrium phenomena are encountered whenever systems are relaxing towards an equilibrium steady state and also whenever systems are driven, i.e., maintained away from equilibrium by external forces. Systems of the driven kind, which are the main focus of this work, cannot be described in general by equilibrium statistical mechanics. Rather, these systems evolve to a nonequilibrium steady state. However, though the statistical mechanics of equilibrium steady states is well understood, analogous general principles to guide the study of steady states far from equilibrium are only just emerging.

Considerable understanding of how microscopic interactions influence the macroscopic properties of nonequilibrium steady states can be gained from the study of interacting particle systems [1, 2] however. These are systems defined on a lattice on which particles hop from site to site, where the precise definition of the stochastic particle dynamics is motivated on physical grounds. Nonequilibrium steady states are then constructed by driving a current of particles (a conserved quantity) through the system. Models of this kind are known as driven diffusive systems [3].

The ongoing interest in driven diffusive systems has been sustained by the variety of the non-trivial behaviour these models exhibit. Even in one dimension, they can undergo phase separation and phase transitions. These transitions may be driven by the boundary dynamics or by defects for example, or, in translationally invariant systems, they may be accompanied by a spontaneously broken symmetry. Furthermore, one observes long-range (i.e. power-law) correlations, not only at criticality, but generically in the steady state of driven diffusive systems. Notably, these properties of driven diffusive systems in one dimension are absent from their one-dimensional classical equilibrium counterparts.

The purpose of this paper is to review recent work on a particular driven diffusive system: the zero-range process (ZRP). In the ZRP, particles hop from site to site on a lattice with a hop rate which depends, most generally, on the site from which it hops and the number of particles at the departure site. Despite being simply stated, it displays all of the non-trivial properties mentioned above, but with the additional virtue that the steady state is given exactly by a factorized form; this simple form of the steady state solution offers an opportunity to analyse these properties exactly.

The ZRP and generalizations we discuss can be employed to investigate fundamental aspects of nonequilibrium statistical mechanics, in that one can address issues such as the role of conservation laws, the range of interactions, constraints in the dynamics, and disorder, all within the framework of an exactly solvable steady state. It has also been widely applied as a model for nonequilibrium phenomena such as sandpile dynamics and the dynamics of avalanches, granular systems, interface growth, polymer dynamics, various transport processes and glasses [4].

The ZRP was previously reviewed in [4]. Since then this model has stimulated considerable interest and a number of authors have contributed to a large body of results. Here we aim to summarize this progress, with emphasis on developments in the ‘physics literature’. In particular, a deeper understanding of condensation—a transition to a phase in which a single site contains a finite fraction of particles in the system—has emerged. It has now become apparent that this condensation transition appears in a number of unexpected contexts

such as wealth condensation in macroeconomies [5], jamming in traffic [6, 7], coalescence in granular systems [8, 9], gelation in networks [10, 11]. Further, we discuss the dynamics of the process by which a condensate emerges, which have recently become better understood. Finally, a number of generalizations of the ZRP which retain the property of a factorized steady state have been identified and we review these generalizations here. We also discuss some results that have not so far appeared in the physics literature such as the relation between condensation in the canonical and grand canonical ensembles (section 4.3.2), dynamics which generate different ensembles (section 6.4) and the Misanthrope process (section 6.2).

The layout of the review is as follows. In section 2 we define the model and derive the steady state. In section 3 we provide details of some of the nonequilibrium phenomena which have recently been modelled using the ZRP. We discuss the homogeneous ZRP (i.e., site-independent hop rates) in section 4 and show how the model undergoes a condensation transition and we compare the analysis of the condensation in the canonical and grand canonical ensembles. We then discuss the coarsening process which determines the dynamics of the condensation. We turn to condensation in the heterogeneous ZRP (i.e., site-dependent hop rates) in section 5 and show how the mechanism of condensation and the coarsening dynamics differ from the homogeneous ZRP. In section 6 we discuss several generalizations of the ZRP, all of which are still characterized by the exact, factorized steady state. These include generalizations to two species of conserved particles (i.e., two conservation laws), to a model in which the hop rates depend on the number of particles at both the departure and the target sites, to continuous mass variables, to allow arbitrary fractions of the mass at a site to hop, to a parallel update mechanism, and to relaxing a conservation law by allowing particle number and/or site number to fluctuate. We summarize in section 7.

2. Definition and steady state

2.1. Definition

The zero-range process is a model in which many indistinguishable particles occupy sites on a lattice. Each lattice site may contain an integer number of particles and these particles hop between neighbouring sites with a rate that depends on the number of particles at the site of departure.

In one dimension, the ZRP is defined on a lattice containing L sites, labelled $l = 1, \dots, L$, and we consider periodic boundary conditions (i.e., site $L+1 =$ site 1). The number of particles at site l is n_l , an integer greater than or equal to zero. The total number of particles in the system is N and the particle density, ρ , is given by

$$\rho = \frac{N}{L}. \quad (1)$$

We consider totally asymmetric dynamics (although this is generalized in section 2.3) such that a particle hops from site l to its nearest neighbour site to the right (site $l+1$) with rate $u(n_l)$.

In general, one can consider any lattice in any dimension, including disordered lattices, and it turns out that one can still obtain the exact steady state (see section 2.3). For simplicity, we will begin by considering the zero-range process in one dimension.

2.1.1. Mapping to an asymmetric exclusion process. A useful property of the zero-range process in one dimension is that it can be mapped onto an exclusion process (i.e., a model in which lattice sites are either occupied by a single particle or they are vacant). This mapping is illustrated in figure 1. The mapping is constructed by identifying a particle configuration in

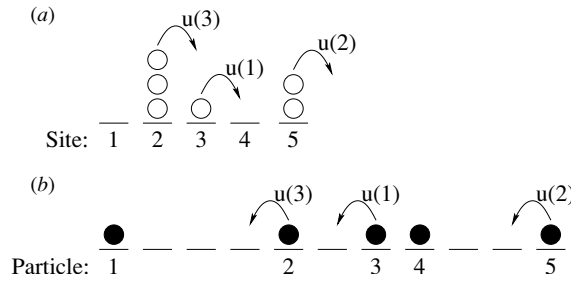


Figure 1. Mapping between the zero-range process and the asymmetric exclusion process.

the ZRP with a corresponding configuration of particles in an exclusion model. (The mapping is unique up to translations of the exclusion process lattice.) To do this, one thinks of particles in the ZRP as vacancies in the exclusion process, and sites in the ZRP as occupied sites in the exclusion process. Thus, in figure 1, site 1 in the ZRP becomes particle 1 in the exclusion process. The next three vacancies in the exclusion process represent the particles at site 2 in the ZRP and then site 2 itself is represented by particle 2 in the exclusion process, and so on. In this way, one obtains an exclusion model on a lattice containing $L + N$ sites and L particles.

The exclusion process dynamics are inferred from the way in which configurations evolve when the corresponding ZRP configurations evolve under the ZRP dynamics: the hop rates in the ZRP, which depend on the number of particles at the departure site, become hop rates in the exclusion process which depend on the distance to the next particle in front. Thus, depending on the form chosen for $u(n)$, there may be a long-range interaction between the particles in the exclusion process.

We remark that this mapping relies on the preservation of the order of particles under the exclusion process dynamics, therefore it is only really useful in one dimension.

2.2. Solution of the steady state

One of the most important properties of the ZRP is that its steady state is given by a simple factorized form. This means that the steady state probability $P(\{n_l\})$ of finding the system in a configuration $\{n_l\} = n_1, n_2, \dots, n_L$ is given by a product of (scalar) factors $f(n_l)$ (one factor for each site of the system) i.e.,

$$P(\{n_l\}) = Z_{L,N}^{-1} \prod_{l=1}^L f(n_l), \tag{2}$$

where $Z_{L,N}$ is a normalization which ensures that the sum of probabilities for all configurations containing N particles is equal to 1, hence

$$Z_{L,N} = \sum_{\{n_l\}} \prod_{l=1}^L f(n_l) \delta\left(\sum_{l=1}^L n_l - N\right). \tag{3}$$

Here, the δ -function has been introduced to guarantee that we only include those configurations containing N particles in the sum. Finally, the factors $f(n_l)$ are determined by the hop rates:

$$f(n) = \prod_{i=1}^n u(i)^{-1} \quad \text{for } n > 0, \quad f(0) = 1. \tag{4}$$

We now turn to the proof of the steady state (2) to (4). The first step is to write the steady state condition that is satisfied by the probabilities $P(\{n_l\})$. This condition balances

the probability current due to hops into a particular configuration with the probability current due to hops out of the same configuration, hence

$$0 = \sum_{l=1}^L [u(n_{l-1} + 1)P(\dots, n_{l-1} + 1, n_l - 1, \dots) - u(n_l)P(\{n_l\})]\theta(n_l), \quad (5)$$

where the Heaviside function $\theta(n)$ is included to emphasize that site l must be occupied for there to be associated hops out of and into the configuration $\{n_l\}$. The next step is to substitute the factorized form (2) into (5) and look to equate each term in the sum separately, hence

$$u(n_{l-1} + 1)f(n_{l-1} + 1)f(n_l - 1) = u(n_l)f(n_{l-1})f(n_l), \quad (6)$$

after cancelling common factors. This equation in turn implies that

$$u(n_{l-1} + 1)\frac{f(n_{l-1} + 1)}{f(n_{l-1})} = u(n_l)\frac{f(n_l)}{f(n_l - 1)} = \text{constant}, \quad (7)$$

for all values of l . The constant can be set equal to unity without loss of generality, hence

$$f(n_l) = \frac{f(n_l - 1)}{u(n_l)}, \quad (8)$$

which is readily iterated to yield (4) where we have set $f(0) = 1$, again without loss of generality. This completes the proof of the steady state (2) to (4).

2.2.1. Some useful general results. From the three equations (2)–(4) we can, in principle, compute all the steady state properties of the ZRP. We remark here that the steady state behaviour is determined by the form of $f(n)$. We could, if we desired, choose *any* form we like for $f(n)$ and then infer the hop rates from

$$u(n) = \frac{f(n - 1)}{f(n)}, \quad (9)$$

having rearranged (8). The consequence of this is that any model, with integer site variables and a conservation law, which has a factorized steady state can be analysed within the framework of a corresponding ZRP—one can use the ZRP to provide a complete account of all the possible steady state behaviour in such models.

It is important to note that $f(n)$, the single-site weight, is distinct from $p(n)$, the probability that a given site contains n particles, given by

$$p(n) = f(n)\frac{Z_{L-1, N-n}}{Z_{L, N}}. \quad (10)$$

To obtain this equation we fix the occupation of site 1 to be n then sum (2) over allowed occupations of the remaining sites subject to the constraint that the remaining number of particles is $N - n$

$$p(n) = \sum_{n_2, \dots, n_L} P(n, n_2, \dots, n_L)\delta\left(\sum_{l=2}^L n_l - (N - n)\right).$$

Thus the δ -function constraint in (3) is important in that it induces correlations between sites.

It is also straightforward to obtain an expression of the mean hop rate $\langle u(n) \rangle$ where $\langle \dots \rangle$ denotes an average in the steady state. In the case of asymmetric dynamics $\langle u(n) \rangle$ is just the particle current. In general,

$$\begin{aligned} \langle u(n) \rangle &= \frac{1}{Z_{L, N}} \sum_{n_1, \dots, n_L} u(n) \prod_{l=1}^L f(n_l) \delta\left(\sum_{l=1}^L n_l - N\right), \\ &= \frac{Z_{L, N-1}}{Z_{L, N}}, \end{aligned} \quad (11)$$

where we have used (8). Thus the mean hop rate is independent of location in the system so it is a conserved quantity. This result is not necessarily obvious in the case where the dynamics are symmetric, for example, in which case $\langle u(n) \rangle$ remains finite although the current vanishes.

Another useful exact result is a recursion for the partition function $Z_{L,N}$,

$$Z_{L,N} = \sum_{n=0}^N f(n) Z_{L-1,N-n}, \quad (12)$$

which can be easily obtained by summing (10) over n . This result is usefully employed as an algorithm to be iterated on a computer. Thus it is straightforward to compare analytic results with numerics for $p(n)$ or $\langle u(n) \rangle$ for example.

2.3. Generalization to an arbitrary lattice

In this section, we generalize the derivation of the steady state given in the previous section to the case where the ZRP is defined on an arbitrary lattice. We use the term arbitrary lattice in a very general sense to mean any lattice in any dimension which may include heterogeneity in connectivity, or in the hop rates.

The steady state still factorizes if the hop rate from site l to k , $u_{kl}(n_l)$, has the general form

$$u_{kl}(n_l) = u_l(n_l) W_{kl}, \quad (13)$$

where $u_l(n_l)$ is a site-dependent function, giving the total rate at which a particle leaves site l if l is occupied by n_l particles, and W_{kl} is the probability that the particle hops to site k . These probabilities define a stochastic matrix for a single particle moving on a finite collection of L (in this context, L is the total number of sites in the system) sites, therefore

$$\sum_k W_{kl} = 1, \quad (14)$$

by conservation of probability.

The matrix W_{kl} defines an arbitrary connectivity of the underlying lattice: if W_{kl} is zero then there is no bond from site l to k across which particles may hop. Moreover, if $W_{kl} \neq W_{lk}$ the hopping is not symmetric, thus the matrix W_{kl} encodes the symmetry of the hopping dynamics. Finally, we can use W_{kl} or $u_l(n_l)$ to include heterogeneity in the hopping dynamics. Thus the definition of the arbitrary lattice implies these three properties (connectivity of the lattice, symmetry and heterogeneity).

The steady state is still given by (2) but with the factors $f(n_l) \rightarrow f_l(n_l)$ now site dependent. These factors are given by

$$f_l(n_l) = \prod_{i=1}^{n_l} \left[\frac{s_l}{u_l(i)} \right] \quad \text{for } n_l > 0, \quad f_l(0) = 1, \quad (15)$$

where the s_l are the steady state weights of a single random walker which moves on a lattice with rates W_{kl} . These weights satisfy

$$s_l = \sum_k s_k W_{lk}. \quad (16)$$

The proof of the steady state follows the same steps as the proof given in the previous section. The steady state condition on the probabilities $P(\{n_l\})$ given by (5) is now

$$0 = \sum_{l=1}^L \left[\sum_{k \neq l} u_k(n_k + 1) W_{lk} P(\dots, n_k + 1, \dots, n_l - 1, \dots) - u_l(n_l) P(\{n_l\}) \right] \theta(n_l), \quad (17)$$

where we have used (14) in the second term on the rhs. Again, we substitute the steady state form (2) (but now with $f(n_l)$ replaced by $f_l(n_l)$) into this equation and look to equate each term in the sum separately, hence

$$u_l(n_l) f_k(n_k) f_l(n_l) = \sum_{k \neq l} u_k(n_k + 1) W_{lk} f_k(n_k + 1) f_l(n_l - 1), \quad (18)$$

for $n_l > 0$. Finally, inserting (15) yields the condition (16).

As an example, consider a one-dimensional chain. One can use the W_{kl} to encode partial asymmetry in the hop rates. That is, if a particle hops to the right with rate $u_{l+1l}(n) = pu(n)$ and to the left with rate $u_{l-1l}(n) = qu(n)$, then this corresponds to the choice

$$W_{l+1l} = p \quad \text{and} \quad W_{l-1l} = q. \quad (19)$$

With this choice, the steady state weights, s_l , satisfy

$$(p + q)s_l = ps_{l-1} + qs_{l+1}, \quad (20)$$

with the solution $s_l = \text{constant}$ for all l . This constant can be taken equal to 1 without loss of generality. Thus the steady state weights (4) are unmodified by the degree of symmetry of the hopping dynamics. Other lattices where $s_l = \text{constant}$ are hypercubic lattices in any dimension and the fully connected geometry. A consequence of this is that if the steady state factorizes for a model in one dimension it will factorize in any higher dimension.

As an example where s_l is not constant we consider symmetric hopping on a body-centred cubic lattice. In this case, each corner site is connected to 6 other corner sites and 8 body-centred sites. Each body-centred site is connected to 8 corner sites. Now, assuming translational invariance, the steady state weights, s_c for corner sites and s_b for body-centred sites, satisfy

$$s_b = \frac{8}{14}s_c. \quad (21)$$

Thus the steady state weights (15) can be written as

$$f_c(n_l) = \prod_{i=1}^{n_l} \left[\frac{7}{u_l(i)} \right], \quad (22)$$

for corner sites and

$$f_b(n_l) = \prod_{i=1}^{n_l} \left[\frac{4}{u_l(i)} \right], \quad (23)$$

for body-centred sites. Then the steady state probabilities are

$$P(\{n_l\}) = Z_{L,N}^{-1} \prod_{l \in C} f_c(n_l) \prod_{l \in B} f_b(n_l), \quad (24)$$

where C denotes the set of corner sites and B denotes the set of body-centred sites.

More complicated s_l may result from disordered hop rates, for example a one-dimensional ZRP with $W_{l+1l} = p_l$ and $W_{l-1l} = q_l$ has been studied in [6, 12–14].

2.4. Mathematical results

The ZRP has been of interest to the mathematical physics community for a long time. It was first introduced by Spitzer [1] in 1970 and has subsequently received a lot of attention in the mathematical literature. Of particular interest is the hydrodynamic limit. Although this is not a limit we focus on, in this section we attempt to give a brief overview of some of this work.

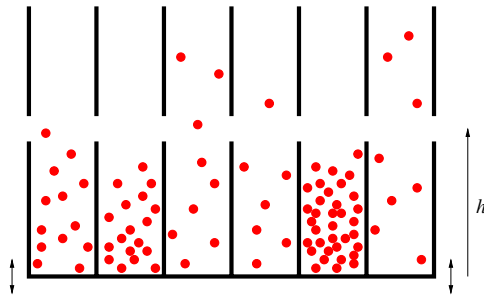


Figure 2. An illustration of the vertically driven container divided into $L = 6$ compartments which are connected by holes at height h . The system has periodic boundary conditions, so particles which leave the right-hand compartment to the right enter the left-hand compartment from the left, and vice versa.

Spitzer [1] obtained the invariant measures for the ZRP for general rates $u(n)$ for finite L . Subsequently, these measures were shown to be invariant even in the case of infinite L [15]. The hydrodynamic limit (i.e., a continuum limit yielding an equation for the time evolution of a coarse-grained particle density field) was obtained, for the one-dimensional symmetric ZRP in [16], and for the one-dimensional asymmetric ZRP in [17, 18], in which case the hydrodynamic equation assumes the form of a nonlinear diffusion equation. The hydrodynamic limit in higher dimensions was given in [19, 20]. For site-dependent rates u_i , without any dependence on occupation number, the existence of invariant measures and the hydrodynamic limit have been proven in [13, 21].

Steady state fluctuation properties of the homogeneous asymmetric ZRP in one dimension were analysed by deriving macroscopic evolution equations for a tagged particle [22] and it was shown that the motion of such a particle is determined by the characteristic lines of the hydrodynamic equation [23]. The size of the largest cluster, in cases where a finite fraction of particles in the system accumulate at a single site, was given in [24], where certain conditions on the hop rates were given to determine whether this fraction is less than 1 or equal to 1 (in which case, the system excluding the cluster site contains a finite number of particles in the limit of infinite system size).

This summary is far from exhaustive. In particular, we have omitted consequences of the hydrodynamic limit, such as the existence of central limit theorems and results for large deviations. But most of these results derive from the invariant measure in the limit of infinite system size (i.e., $L \rightarrow \infty$ then $t \rightarrow \infty$) whereas our approach is to compute the steady state on the finite lattice and only then consider the thermodynamic limit (i.e., $t \rightarrow \infty$ then $L \rightarrow \infty$).

3. Applications

3.1. Shaken granular gases

A variety of granular systems can be related to zero-range processes, for example models of sandpile dynamics [25, 26] or models of mass transport through a series of traps [27]. Another example, which has received a lot of attention more recently, is models of shaken granular gases. These models are based on experiments in which a container is divided into L equal compartments by walls, where each wall contains a narrow horizontal slit at height h . The container is then mounted on a shaker and filled with N particles, e.g., plastic balls or sand. See figure 2 for an illustration of the experimental set-up. When the system is shaken vertically, the particles hop from one compartment to another.

The dynamics of this system clearly resemble those of the zero-range process. To make the correspondence explicit, one has to find an appropriate form for the ZRP hop rates $u(n)$ motivated by the kinetics of particles within a compartment. Two forms for $u(n)$ have been considered [8, 28] and they have been compared, in the context of the ZRP, in [29].

The first of these approaches is due to Eggers [8]. An expression for $u(n)$ is obtained by solving the equations of motion found using the kinetic theory of vertically shaken granular gases. For two-dimensional circular discs, the result is

$$u(n) = u_0 n^2 e^{-a(n/N)^2}, \quad (25)$$

where u_0 is a constant (which only sets the time scale), n is the number of particles in the compartment and a , given by

$$a = 4\pi g r^2 (1 - e)^2 \frac{h}{A^2 f^2} \left(\frac{N}{\Omega L} \right)^2, \quad (26)$$

depends on the system parameters: g , the acceleration due to gravity; r , the radius of the discs; e , the coefficient of restitution; A , the amplitude of the driving; f , the frequency of the driving; and Ω , the width of each compartment.

For certain values of a the system is found to evolve to a steady state in which a single compartment contains most of the particles. In the thermodynamic limit, $N \rightarrow \infty$ with L fixed, this corresponds to a phase in which the fraction of particles in a single compartment is 1. For $L = 2$ this is a second-order transition, from a phase in which particles are homogeneously distributed amongst the compartments, and is accompanied by a spontaneously broken symmetry. The model was originally introduced for $L = 2$, but has subsequently been studied for $L = 3$ [30] and arbitrary L [31, 32], where it was shown that the transition is first order for $L > 2$. The dynamics of the coarsening leading to the condensed phase was studied in [9]. Experiments and simulation indicate that the system evolves to its steady state in the condensed phase in two stages: firstly, most of the particles cluster in a few compartments, then these clusters disappear one by one until only a single cluster remains. The mean cluster size is found to grow like $[\ln(t)]^{1/2}$. The appropriate form of the hop rates for a bidisperse granular gas was derived in [33]; this model corresponds to a ZRP with two different types of particles, a generalization of the ZRP we discuss in section 6. Finally, an asymmetric case where a current of particles is supported due to holes connecting compartments at different heights has been studied [34]. In this case, the ZRP description does not yield a factorized steady state since the hop rates do not fall into the class discussed in section 2.3 but the assumption of factorization provides a mean-field-type approximation.

The second approach is due to Lipowski and Droz [28], who propose a simpler model than that of Eggers which still captures the essence of the phenomena. They consider a hop rate of the form

$$u(n) = \frac{n}{N} \exp\left(-\frac{1}{T_0 + \Delta(1 - n/N)}\right), \quad (27)$$

where T_0 and Δ are positive constants, which depend on system parameters. The basis of Eggers' derivation of (25) was that the effective temperature of a granular system decreases when the particle density increases. The form (27) is the simplest hop rate which reproduces this fact, due to the linear n -dependence in the denominator of the exponential. The choice (27) then describes dynamics under which a particle is selected at random and moved to a randomly chosen neighbouring compartment with probability $\exp(-1/[T_0 + \Delta(1 - n/N)])$.

As in Eggers' model, this model undergoes a transition between a homogeneous phase and a phase in which most of the particles occupy a single compartment. Again, the transition is second order for $L = 2$ [28, 35] and first order for $L > 2$ [36, 37]. During the coarsening stage of the dynamics the mean cluster size is found to grow like $\ln(t)$ [28, 29, 35–37].

3.2. Networks

A network is defined as a set of nodes interconnected by links—mathematically a network is simply a graph and the links, which may be directed or undirected, are the edges of the graph. The interest within the statistical physics community into networks has been with regard to the statistics of the connectivity properties. For example, the degree of a node is the number of links attached to it and the degree probability distribution may exhibit a ‘fat tail’ or even a power-law asymptotic behaviour. Simple models to generate dynamically such networks and statistics have been introduced and studied extensively (for recent reviews, see [38, 39]).

3.2.1. Growing and rewiring networks. The dynamics of growing networks is introduced through stochastic rules for attaching links to nodes. Basically, there are two types of dynamical networks: growing networks where links and nodes are added to the networks [40] and ‘rewiring’ or ‘equilibrium’ networks where the number of nodes and links is fixed but links re-attach stochastically from one node to another [41, 42].

In growing networks, power-law degree distributions may be generated through preferential attachment. The idea is to add a new node to the network at each time step and to attach this node to an existing node selected with probability proportional to its degree. It has been shown that the linear dependence of attachment rate on degree gives rise to a power-law degree distribution [40]. If the attachment rate grows more slowly than linearly in the degree of the node one can generate stretched exponential degree distributions; if the attachment rate grows faster than linearly in the degree of the node, a condensate node is generated which is connected to nearly all other nodes [44, 10].

Generally, condensation (also referred to as gelation in the network literature) occurs when a node captures a finite fraction of the total number of links. As well as condensation induced by nonlinear preferential attachment one can have condensation induced by heterogeneity [11]. Here nodes have an intrinsic fitness and the rate of attachment is proportional to this. The two types of condensation are reminiscent of the two types of condensation seen in the ZRP to be discussed in sections 4.2 and 5.1. The connection is explicit in the case of rewiring networks which we review in more detail in section 6.2.4. In these networks the nodes correspond to sites of the lattice and the dynamics of links becomes equivalent to the dynamics of the particles in a generalized ZRP on a fully connected geometry.

3.2.2. Directed edges and two-species ZRP. One can also consider networks where the edges are directed. Then each node is characterized by its in-degree and out-degree which are the number of edges pointing into the node and the number of edges pointing out of the node, respectively. Rewiring dynamics consists of a rate for rewiring the outgoing end of an edge and a rate for rewiring the ingoing end of an edge [42] and these rates most generally depend on the in- and out-degree of both the departure and destination nodes.

If we think of ingoing edges and the outgoing edges as two species of particles we have a generalization of the two-species ZRP discussed in section 6.1. Again the model is defined on a fully connected geometry and one obtains a condition for factorization similar in form to the factorization condition for the two-species ZRP [42]. It should be possible to analyse condensation transitions within this framework.

3.3. Coarse-grained descriptions

So far we have presented applications of the ZRP which essentially involve mapping various models on to the ZRP at the level of exchange of particles between sites of the system. More

generally, however, one may think of the sites of the ZRP as representing domains of some driven system—this is a most natural picture within the exclusion process interpretation of the ZRP (figure 1). The domains may have some internal structure, for example further degrees of freedom, but all this is integrated out, and one is left with an effective dynamics of exchange of length between domains given by $u(n)$.

3.3.1. Bus-route model. An early example of the use of the ZRP as an effective coarse-grained description of more complicated microscopic dynamics is the ‘bus-route model’ [45]. The model is defined on a $1d$ lattice. Each site (bus stop) is either empty, contains a bus (a conserved particle) or contains a passenger (non-conserved quantity). The dynamical processes are that passengers arrive at an empty site with rate λ ; a bus moves forward to the next stop with rate 1 if that stop is empty; if the next stop contains passengers the bus moves forward with rate β and removes the passengers. The bus-route model can be related to the ZRP by an approximation of a mean-field nature in which we integrate out the non-conserved quantity (passengers). The bus-route dynamics can be thought of as an exclusion process which is mapped onto ZRP dynamics as in figure 1: the buses correspond to the sites of the ZRP and the number of bus stops between a bus and the next bus along the route corresponds to the number of particles at the site of the ZRP. The idea is that the hop rate of the buses becomes a function of the distance to the next bus ahead. Now, the mean time elapsed since a bus stop was last visited is given by n/v where n is the distance to the next bus ahead and v is the steady state speed. Therefore the mean-field probability that the site next to a bus is not occupied by a passenger is $\exp(-\lambda n/v)$. From this probability an effective hop rate for a bus into a gap of size n is obtained by averaging the two possible hop rates 1, β :

$$u(n) = \beta + (1 - \beta) \exp(-\lambda n/v). \quad (28)$$

This example serves to illustrate how local dynamics may generate an effective ZRP dynamics defined by $u(n)$.

The phenomenon of condensation, wherein a single site of the ZRP contains a finite fraction of the density (to be discussed in 4.2), would correspond to a finite fraction of the bus stops being between two buses, i.e. there is a jam of buses in the steady state and all the buses arrive at a bus stop at once! However, it turns out that since $u(n)$ decays exponentially the condition for a strict phase transition in the thermodynamic limit is not met, unless we take the passenger arrival rate $\lambda \rightarrow 0$. However, on any *finite* system for λ sufficiently small, an apparent condensation will be seen.

In [45] a dual model to the bus-route model was used to model clogging in pipes. Also a model of ant trails introduced in [46] may be mapped exactly onto the bus-route model.

3.3.2. Phase separation in one-dimensional systems. Recently, using the ZRP as the effective description has allowed insight into the long-standing question of phase separation in one-dimensional driven systems [47]. Within this description phase separation is manifested by the emergence of one large domain and this corresponds to the phenomenon of condensation in the ZRP. The criterion for phase separation corresponds to the criterion for condensation within the ZRP [47, 48].

The idea is that domains of particles on a one-dimensional lattice can be represented by the sites of a ZRP and the dynamics by which domains exchange length can be described by $u(n)$. Thus, in this context, $u(n)$ corresponds to a conserved current flowing out of a domain of size n and the assumption is that domains are uncorrelated in that the current only depends on the length of a domain. The approach reduces a many-domain problem to the properties of a single domain; the key step is the identification of the effective current $u(n)$.

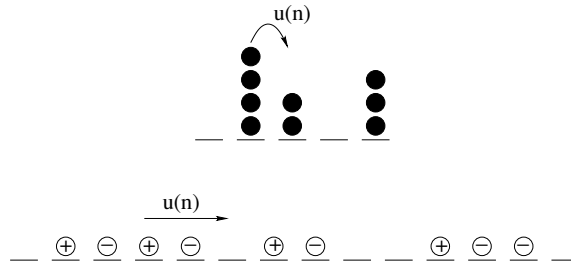


Figure 3. A microscopic configuration of a two-species driven model (bottom) and its corresponding configuration in the ZRP (top).

As we shall see in section 4.4 the criterion to have condensation is that either the hop rates $u(n)$ in the ZRP should decrease to 0 as n increases or else $u(n)$ should decay more slowly to an asymptotic value β than $\beta(1 + 2/n)$. Thus, the criterion for phase separation to occur is that the current of particles out of a domain of size n should either decay to zero with domain size or decay to a non-zero value more slowly than $\beta(1 + 2/n)$.

In the development of this approach particular attention has been focused on a class of exclusion models with positive particles, negative particles and vacancies [49]. The model we consider is defined on a one-dimensional ring of L sites. Each site i is associated with a ‘spin’ variable s_i . A site can either be vacant ($s_i = 0$) or occupied by a positive ($s_i = +1$) or a negative ($s_i = -1$) particle. Particles are subject to hard-core repulsion and a nearest-neighbour ‘ferromagnetic’ interaction, defined by the potential

$$V = -\frac{\epsilon}{4} \sum_i s_i s_{i+1}. \tag{29}$$

Here $0 \leq \epsilon < 1$ is the interaction strength, and the summation runs over all lattice sites. The model evolves according to the nearest-neighbour exchange rates

$$+ - \xrightarrow{(1+\Delta V)} - + \quad - + \xrightarrow{q(1+\Delta V)} + - \quad +0 \xrightarrow{\alpha} 0+ \quad 0- \xrightarrow{\alpha} -0, \tag{30}$$

where ΔV is the difference in the potential V between the initial and final states. One can think of this as a generalization of the Katz–Lebowitz–Spohn driven lattice gas (KLS model) to be discussed in section 6.2.2.

The sites of the exclusion process correspond to the vacancies in this model and the domains to the consecutive sequences of positive and negative particles as illustrated in figure 3. Note that although a domain comprises both positive and negative particles, the effective description is in terms of a one-species ZRP. In the case $q > 1$ (where positive particles drift preferentially to the left of negative particles) the steady state comprises three pure regions of vacancies, positive particles and negative particles. This corresponds to strong condensation where one site of the ZRP holds a fraction tending to one of the particles. The current of particles is exponentially small in the system size [50] and $u(n)$ decays exponentially to zero. A more subtle question is whether condensation and phase separation can occur for $q < 1$.

In the case $\epsilon = 0$ the model reduces to that studied in [51, 52] and an exact solution is available via a matrix product ansatz [53]. The mapping of the steady state of the exclusion process to the steady state of a ZRP becomes exact and allows an exact identification of the effective current $u(n)$. In the case of equal densities of positive and negative particles the function $u(n)$ is precisely equal to the current flowing in the steady state of an exclusion process of length n with open boundary conditions [47]. This gives a simple recipe for determining

$u(n)$ and provides a numerical test for phase separation, even when the open system cannot be solved exactly [47]. In the case of nonequal densities, the current is given by an exclusion process of length n with periodic boundary conditions but with non-conservation of the charge of particles [48]. In either case the current decays with domain size as $u(n) = u(\infty)(1+3/(2n))$ which does not satisfy the condensation criterion. However, one may have very strong finite size effects leading to sharp crossover effects which may give the impression of a phase transition on finite systems [54, 55].

More generally, when $\epsilon \neq 0$, and the ferromagnetic interaction is switched on, no exact solution is available. However, using the identification of $u(n)$ suggested by the $\epsilon = 0$ case predicts that phase separation (for $q < 1$) should occur when ϵ is sufficiently large [56]. This effect is enhanced when the densities of positive and negative particles are unequal [48]. Although the mapping to a ZRP is not expected to be exact when $\epsilon \neq 0$, its predictions have proven accurate so far.

4. Homogeneous system

We now return to the ZRP defined in section 2. In this section, we study the homogeneous system in which hopping rates $u(n)$ and therefore steady state weights $f(n)$ are site independent (see section 2.2). We will focus on the phenomenon of condensation. As we shall see the condition for this to occur depends on the asymptotic behaviour of $f(n)$ and therefore $u(n)$. We also compare the analysis of condensation in the canonical and grand canonical ensembles.

As we have seen in the introduction and section 3 condensation is observed in a variety of physical contexts. The factorization property allows us to analyse exactly the condensation mechanism within the ZRP. The first analysis of this kind on a homogeneous system was carried out in [57] where the grand canonical partition function of a ‘balls-in-boxes’ model is analysed. It should be noted that condensation also occurs in other systems that cannot be solved exactly and a factorized steady state becomes an approximation within which to study condensation [58, 59].

Ideally, one would wish to demonstrate condensation by analysing the distribution of the number of particles $p(n)$ given by (10). This expression involves $Z_{L,N}$, defined by (3), which is the equivalent of the canonical partition function since the number of particles is held fixed. However, to understand the condensation phenomenon, it is simplest to work within the grand canonical ensemble where the particle number fluctuates.

4.1. Grand canonical ensemble

We define the grand canonical partition function as

$$\mathcal{Z}_L(z) = \sum_{n=0}^{\infty} z^n Z_{L,n}, \quad (31)$$

where z is the fugacity which is chosen to fix the density ρ through

$$\rho = \frac{z}{L} \frac{\partial \ln \mathcal{Z}_L(z)}{\partial z}. \quad (32)$$

Now, using (3), we have

$$\mathcal{Z}_L(z) = \sum_{\{m_l=0\}}^{\infty} z^{\sum_l m_l} \prod_{l=1}^L f(m_l) = [F(z)]^L \quad (33)$$

where

$$F(z) = \sum_{m=0}^{\infty} z^m f(m) \tag{34}$$

and

$$\rho = z \frac{F'(z)}{F(z)}. \tag{35}$$

One views this condition either as defining explicitly $\rho(z)$, or as defining implicitly $z(\rho)$.

The distribution of the number of particles at a given site becomes

$$p(n) = z^n f(n) \frac{Z_{L-1}}{Z_L} = \frac{z^n f(n)}{F(z)} \tag{36}$$

in the grand canonical ensemble. We can then calculate the mean hop rate $\langle u(n) \rangle = \sum_n p(n)u(n)$ as

$$\langle u(n) \rangle = z, \tag{37}$$

endowing the fugacity with a physical interpretation. We emphasize that here the angle brackets denote a steady state average within the grand canonical ensemble. This is to be compared with the expression of the mean hop rate in the canonical ensemble, given by (11).

4.2. Condensation

It is easy to show, by taking the derivative with respect to z , that the right-hand side of (35) is an increasing function of z . Also let the radius of convergence of $F(z)$, defined in (34), be $z = \beta$. Thus increasing the value of z corresponds to increasing values of the density $\rho(z)$ until z takes its maximum value β which corresponds to the maximal value of the density ρ_c

$$\rho_c = \beta \frac{F'(\beta)}{F(\beta)}. \tag{38}$$

If ρ_c is infinite then for any finite density one can find a solution of (35) for z . On the other hand, if ρ_c is finite then for $\rho > \rho_c$ one can no longer satisfy (35) and we have condensation.

Let us be specific and consider $f(n)$ with large n asymptotic form

$$f(n) \sim \frac{A}{\beta^n n^b}. \tag{39}$$

Then $z = \beta$ is clearly the radius of convergence of (34) and whether ρ_c is finite or not is controlled by $F'(\beta)$ which converges for $b > 2$. Thus for $b > 2$ we will have condensation.

To understand what is happening we look at the single-site distribution $p(n)$ (10) for the number of particles. For $n \gg 1$ this becomes, using the asymptotic form (39),

$$p(n) \simeq \frac{(z/\beta)^n A}{F(z) n^b}. \tag{40}$$

Consider first $b \leq 2$ in which case $\rho_c = \infty$. The distribution (40) is a power law $\sim 1/n^b$ with an exponential pre-factor cutting off the distribution at $n \sim 1/|\ln(z/\beta)|$. Increasing values of z imply that the distribution is cut off at larger values of n and thus correspond to increasing values of the density. If $b \leq 2$ one can obtain any desired density $\rho = \sum_n np(n)$ by choosing z to cut off the power law at suitable n .

For $b > 2$, on the other hand, the power law $1/n^b$ has a finite mean. Thus letting the cut-off tend to infinity (i.e., $z \rightarrow \beta$) only corresponds to a finite density. Therefore for $\rho > \rho_c$ one requires an extra piece to the distribution which represents the condensate. The shortfall in the density, $\rho - \rho_c$, will be made up by the presence of this piece.

To confirm that the condensate corresponds to a single site we consider its canonical weight: if there is a single condensate site the weight will be $Lf(N - N_c) = \mathcal{O}(L^{1-b})$, whereas if the condensate is spread over two sites the weight will be $\binom{L}{2} \sum'_n f(N - N_c - n) f(n) = \mathcal{O}(L^{3-2b})$, where the prime indicates a sum over terms $n = \mathcal{O}(N)$. The factors L and $\binom{L}{2}$ come from the number of ways of choosing the condensate site(s). Now the condition for condensation, $b > 2$, implies that $1 - b > 3 - 2b$, therefore in the limit $L \rightarrow \infty$ configurations with a single condensate are dominant over configurations in which the condensate is spread over two (or more) sites. Note that in the condensed phase we have used canonical weights, because in this phase the equivalence between ensembles breaks down as we discuss in the next subsection. A rigorous proof that the condensate corresponds to a single site is given in [60].

4.3. Relation between canonical and grand canonical partition functions

We now return to the calculation of the canonical partition function and the question of equivalence between canonical and grand canonical ensembles.

We may write $Z_{L,N}$ using Cauchy's theorem as

$$Z_{L,N} = \oint \frac{ds}{2\pi i} s^{-(N+1)} \mathcal{Z}_L(s) = \oint \frac{ds}{2\pi i} s^{-(N+1)} [F(s)]^L, \quad (41)$$

where the integral is around a closed contour about the origin in the complex s plane. For large N, L (41) is dominated by the saddle point of the integral which we denote by $s = z$. Defining

$$\phi(s) = -\rho \ln s + \ln[F(s)] \quad (42)$$

the saddle point z is given by $\phi'(z) = 0$ and this recovers precisely (35). Moreover,

$$Z_{L,N} \simeq \frac{1}{(2\pi L)^{1/2}} \frac{1}{|\phi''(z)|^{1/2}} \frac{e^{L\phi(z)}}{z} \quad (43)$$

$$= \frac{1}{(2\pi L)^{1/2}} \frac{1}{|\phi''(z)|^{1/2}} \frac{\mathcal{Z}_L(z)}{z^{N+1}}. \quad (44)$$

Thus when one has a valid saddle point the canonical and grand canonical ensembles are equivalent [61]. However, for a valid saddle point the value of z must be within the radius of convergence of $F(z)$ since the integral contour cannot be extended to enclose the singularity at $s = \beta$. Thus for $\rho > \rho_c$ the asymptotics of the canonical partition function must take on an alternative form, signalling the phase transition. Only recently has a detailed analysis of this form and the condensation mechanism been carried out [62]; we give a brief discussion here and refer the reader to [62]. Also note that the equivalence of ensembles has been studied in [60]. There it was shown that for $\rho > \rho_c$ the fluid part of the canonical distribution coincided with the critical grand canonical distribution.

4.3.1. Condensation within canonical ensemble. As $\rho \nearrow \rho_c$ the saddle point z approaches a branch cut singularity at $z = \beta$. For $\rho \geq \rho_c$ there is no valid saddle point, but the integral (41) is still dominated by the region $s \simeq \beta$. Thus in the critical region and in the whole of the condensed phase one can expand about $s = \beta$. For $f(n)$ of the form (39) we let $s = \beta(1 - u)$ and expand $F(s)$ for u small as

$$F(\beta(1 - u)) = \sum_{k=0}^{r-1} \frac{(-\beta u)^k F^{(k)}}{k!} + B u^{b-1} + \dots \quad (45)$$

where r is the integer part of b and $F^{(k)} = \frac{d^k}{ds^k} F(s)|_{s=\beta}$ (see, e.g., [63]). The second term in (45) is the leading singular part and the coefficient B is determined through

$$B = \lim_{u \rightarrow 0} \frac{u^{r+1-b}}{(b-1) \cdots (b-r)} \frac{d^r F}{du^r} \tag{46}$$

and it can be shown that

$$B = A\Gamma(1-b), \tag{47}$$

where $\Gamma(x)$ is the usual gamma function.

In the condensed phase, the contour integral in (41) can be deformed to run along the imaginary axis in the complex u plane. Then the asymptotic behaviour is controlled by the nonanalytic term $A\Gamma(1-b)u^{b-1}$ and one obtains

$$Z_{L,N>N_c} \sim \frac{F(\beta)^L}{\beta^N} \int_{-i\infty}^{i\infty} \frac{du}{2\pi i} \exp L \left[u(\rho - \rho_c) + \cdots + u^{b-1} \frac{A\Gamma(1-b)}{F(\beta)} + \cdots \right]. \tag{48}$$

The asymptotics of this integral can be evaluated by wrapping the contour around the negative real axis and one finds that the leading contribution is

$$Z_{L,N>N_c} \simeq \frac{F(\beta)^{L-1}}{\beta^N} \frac{A}{(\rho - \rho_c)^b L^{b-1}} \quad \text{all non-integer } b > 2. \tag{49}$$

Thus we have a distinct supercritical form for the canonical partition function (49) to be compared with the grand canonical form (44) which holds in the subcritical regime. These forms can be used to analyse the piece of $p(n)$ which represents the condensate and one finds distinct universal forms for $2 < b < 3$ and $b > 3$ [62].

4.3.2. Condensation within grand canonical ensemble. In contrast, an analysis of the condensate within the grand canonical ensemble requires allowing $z > \beta$, thus one has to impose cut-offs on the sums [64] and modify (33) to

$$\mathcal{Z}_L(z) = [F_N(z)]^L \quad \text{where } F_N(z) = \sum_{n=0}^N f(n)z^n. \tag{50}$$

Here the cut-off N ensures that no site contains more than N , the total number of particles in the canonical ensemble, but note that the total number of particles in the grand canonical ensemble still fluctuates. Then to satisfy (38) for $\rho > \rho_c$ one takes $z = \beta(1 + \eta)$ with η small and chooses η so that $\beta F'(\beta) < z F'_N(z) = \mathcal{O}(1)$. The result is [65]

$$z = \beta(1 + \eta), \quad \eta = (b-2) \frac{\ln N}{N} + \frac{\ln \ln N}{N} + \mathcal{O}\left(\frac{1}{N}\right), \tag{51}$$

where the coefficient of the $\mathcal{O}(1/N)$ term has to be chosen to ensure that (38) is satisfied. This result implies that the mean hopping rate, $\langle u \rangle = z$ in the grand canonical ensemble, overshoots its thermodynamic limit value β on a large but finite system [65].

However, the condensate is not correctly reproduced by the introduction of a cut-off. This is seen by noting that (36) becomes

$$p(n) = \frac{A}{F(z)} \exp(n\eta - b \ln n) \tag{52}$$

which has a minimum at $n = b/\eta$ and increases to a boundary maximum at the maximum allowed value $n = N$, see figure 5. The piece of $p(n)$ to the right of the minimum corresponds to the condensate. However, this piece is not centred about the value $(\rho - \rho_c)L$ as the condensate should be. This is because we have to take $z > 1$ in order to account for the

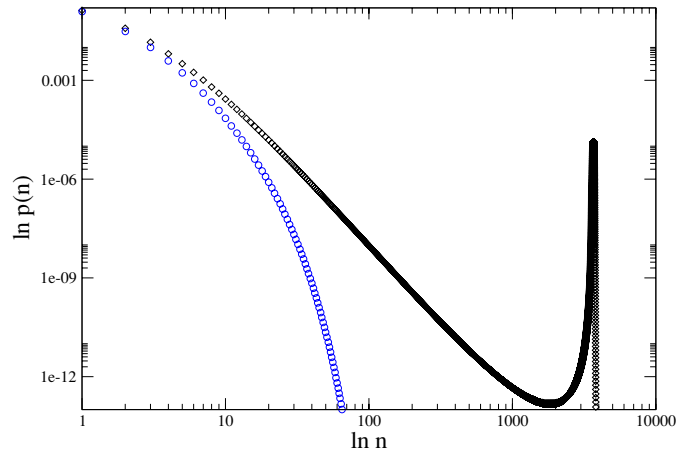


Figure 4. In–ln plot of the single-site distribution $p(n)$ versus particle number n . The data are obtained by iterating the recursion relation (12) for $Z_{L,N}$ for $L = 1000$ and $b = 5$. The circles represent $\rho = 1/4$ where the system is in the fluid phase; the diamonds represent $\rho = 4$ where the system is in the condensed phase.

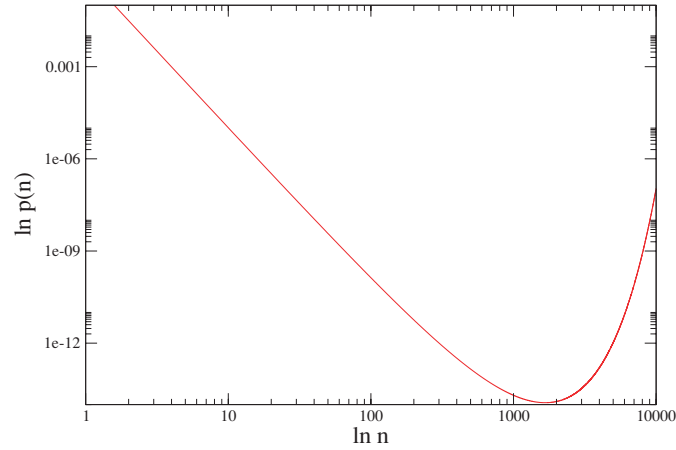


Figure 5. In–ln plot of the grand canonical single-site distribution $p(n)$ given by (52) versus particle number n . The curve shown is for $A/F(z) = 1$, $\eta = 0.002$ and $b = 5$, and should be compared with the diamonds in figure 4.

condensate which implies that $p(n)$ must increase at large n rather than having a maximum (as is seen in figure 4). This is in contrast to usual Bose condensation, in momentum space in the ideal Bose gas, where the condensate is accounted for by taking $z \nearrow 1$ in the appropriate way [61]. To correctly describe the condensate in the present case one has to work within the canonical ensemble [62].

4.4. Condensation condition on hop rates $u(n)$

So far our discussion of the condensation mechanism has centred on the single-site weight function $f(n)$. In most applications of ZRP, however, one is given the hop rate $u(n)$ (see section 3), therefore we now consider the conditions for condensation implied on $u(n)$.

For condensation, we require that the infinite series $zF'(z) = \sum_{n=1}^{\infty} nz^n f(n)$ converges for some value of z . The ratio of successive terms $n - 1, n$ in the series is given by

$$\frac{(n - 1)f(n - 1)}{znf(n)} = \frac{u(n)}{z} \left(1 - \frac{1}{n}\right). \tag{53}$$

The ratio test [66] tells us that for convergence of the series this ratio should decay more slowly than $1 + 1/n$, thus for condensation to occur, i.e. convergence at $z = \beta$, $u(n)$ should decay more slowly than

$$u(n) \simeq \beta \left(1 + \frac{2}{n}\right) \quad \text{for } n \gg 1. \tag{54}$$

This corresponds to the asymptotic expression for $f(n)$ (39) studied in the previous subsections.

If $u(n)$ decays, for example, as

$$u(n) \simeq \beta \left(1 + \frac{a}{n^\lambda}\right) \quad \text{for } n \gg 1, \tag{55}$$

where $0 < \lambda < 1$, one obtains condensation but with the fluid phase now distributed according to a stretched exponential distribution $p(n) \sim \exp(-an^{1-\lambda}/(1-\lambda))$.

If $u(n)$ increases as $n \rightarrow \infty$ the series has infinite radius of convergence and so condensation never occurs. On the other hand if $u(n) \rightarrow 0$ as $n \rightarrow \infty$, the series will have zero radius of convergence. This implies condensation at any density. Moreover, the density in the fluid phase will tend to zero, therefore the fraction of particles in the condensate will tend to 1.

These conditions on $u(n)$ lead directly to the criteria for phase separation discussed in section 3.3.2. When the current of particles out of a domain of size n decays to an asymptotic value β , for large n , more slowly than $\beta(1 + 2/n)$, phase separation occurs above some critical particle density. This phase separated state consists of a single domain containing a finite fraction (less than 1) of particles in the system. On the other hand, when the current of particles out of a domain of size n decays asymptotically to zero, then strong phase separation occurs leading to a state in which a single domain contains a fraction of particles equal to 1—the remainder of the system contains a finite number of particles.

One can also infer the condensation phenomena that occur in the shaken granular gases discussed in section 3.1. In this case, both the hop rates (25) and (27) decay to zero as $n \rightarrow \infty$. Thus, in the limit $N, L \rightarrow \infty$ with ρ fixed, both models will evolve to a steady state in which a single compartment contains a fraction of particles equal to 1 and the remainder of the system contains a finite number of particles. The corresponding analysis in the limit $N \rightarrow \infty$ with L fixed has been carried out in [29].

4.4.1. Explicitly solvable case. It turns out that if one takes $u(n) = \beta(1 + b/n) \forall n > 0$ one can calculate the critical density, given by (38), exactly [60, 67]. In this case, one has

$$f(n) = \beta^{-n} \prod_{i=1}^n \frac{i}{i+b} = \beta^{-n} \frac{n!}{(b+1)_n} \tag{56}$$

where

$$(a)_n = a(a+1) \cdots (a+n-1), \tag{57}$$

is the Pochhammer symbol. Thus

$$F(z) = \sum_{n=0}^{\infty} (z/\beta)^n \frac{n!}{(b+1)_n} = \sum_{n=0}^{\infty} (z/\beta)^n \frac{(1)_n (1)_n}{(b+1)_n n!}$$

$$F'(z) = \frac{1}{\beta} \sum_{n=0}^{\infty} (z/\beta)^n \frac{(n+1)(n+1)!}{(b+1)_{n+1}} = \frac{1}{\beta(b+1)} \sum_{n=0}^{\infty} (z/\beta)^n \frac{(2)_n(2)_n}{(b+2)_n n!}$$

and one finds using the hypergeometric function identity [68]

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for } c > a+b \quad (58)$$

that $F(\beta) = b/(b-1)$ and $F'(\beta) = b/(\beta(b-1)(b-2))$. Thus (38) yields

$$\rho_c = \frac{1}{b-2}. \quad (59)$$

Note, however, that one does not expect ρ_c to be universal, i.e., it generally depends on the form of $f(n)$ for small values of n , not just on the asymptotic behaviour.

4.5. Dynamics of condensation

In this section, we turn to the dynamics of the condensation process. Starting from a homogeneous distribution of particles, the condensate emerges as the result of a non-trivial coarsening process [45]. The excess particles accumulate on a number of sites and the coarsening is then determined by the exchange of particles between these sites; this results in their elimination and hence growth in the mean particle number at such sites. The object of this section is to determine the scaling satisfied by this growth.

We consider the case where the hops rates have the asymptotic form

$$u(n) = 1 + b/n, \quad (60)$$

when the system evolves from an initially random distribution of particles at a supercritical density $\rho > \rho_c$. Two complementary approaches have been taken to analyse the dynamics for this case; one is a mean-field approach, and the other is a heuristic scaling approach based on random walk arguments.

4.5.1. Mean-field dynamics. The mean-field approach is due to Godrèche [67] and is based on earlier studies of the dynamics of urn models [69, 70]. Here we outline this approach, and refer the reader to [67] for details.

In the mean-field system we assume that the steady state factorizes, and we allow particles to hop from any site to any other in the system. Thus the probability $p_t(n)$ that a site contains n particles at time t satisfies a *master* equation given by

$$\frac{dp_t(n)}{dt} = u(n+1)p_t(n+1) + \langle u_t \rangle p_t(n-1) - u(n)p_t(n) - \langle u_t \rangle p_t(n), \quad (61)$$

for $n > 0$, where $\langle u_t \rangle = \sum_{m=1}^{\infty} u(m)p_t(m)$ is the expectation value of the hop rate at time t , and

$$\frac{dp_t(0)}{dt} = u(1)p_t(1) - \langle u_t \rangle p_t(0). \quad (62)$$

These equations are nonlinear, due to the dependence of u_t of $p_t(n)$, however the work of [67, 69, 70] has shown how one can obtain the late-time dynamics in the condensed phase (and also at criticality, although we do not address this point further here). One finds that the system decomposes into two distinct parts: one which includes sites that have effectively reached a steady state at density ρ_c , and the other which includes sites that contain the excess $(\rho - \rho_c)L$ particles that form the condensate.

In the condensed phase, in the limit $t \rightarrow \infty$, $\langle u_t \rangle \rightarrow \langle u \rangle$, the steady state expectation value of the hop rate. In the steady state, $\langle u \rangle$ is equal to the fugacity z , and in the condensed phase z is given by its maximum allowed value, β , determined by the radius of convergence of the sum (34). For the rate (60) $\beta = 1$ therefore $\langle u \rangle = 1$. Therefore for late times one sets

$$\langle u_t \rangle \simeq 1 + A\epsilon_t, \quad (63)$$

where A is an amplitude and ϵ_t is a small time scale, both to be determined later. Next, one observes that the terms in (61) which reduce the number of particles at a site are those which depend on $u(n)$, and those which increase the number of particles at a site depend on $\langle u_t \rangle$. Sites at which n is finite reach an effective steady state and the dynamics of the condensation is determined by those sites at which n is large (we will refer to sites at which n is extensive as condensate sites). For sufficiently late times these sites are in the scaling regime. Thus one defines the scaling variable $x = n\epsilon_t$, and the probabilities $p_t(n)$ are rewritten in terms of a scaling function $g(x)$, i.e.,

$$p_t(n) \simeq \epsilon_t^2 g(x). \quad (64)$$

Now one substitutes the three equations (60), (63) and (64) into the master equation (61), which yields the differential equation

$$g''(x) + \left(\frac{x}{2} - A + \frac{b}{x} \right) g'(x) + \left(1 - \frac{b}{x^2} \right) g(x) = 0, \quad (65)$$

satisfied by $g(x)$. As a byproduct, one also obtains $\epsilon_t \sim \epsilon_t^3$, hence

$$\epsilon_t \sim t^{-1/2}. \quad (66)$$

The amplitude A is determined by the properties of $g(x)$ for small and large x and is found to be universal in the sense that it only depends on b , therefore the scaling function $g(x)$ is also universal in this sense. Properties of the solution of (65) for $g(x)$ can be found in [67, 69].

The picture that emerges from this mean-field analysis is that the typical number of particles at condensate sites scales as $t^{1/2}$. Since the total number of particles at such sites is $(\rho - \rho_c)L$, the number of condensate sites must scale as $(\rho - \rho_c)L^{-1/2}$. Hence the mean condensate size $\langle m(t) \rangle$, defined as the total number of particles at condensate sites divided by the number of condensate sites at time t , obeys a scaling law

$$\langle m(t) \rangle \sim t^\delta, \quad (67)$$

defining the exponent δ , where

$$\delta = 1/2. \quad (68)$$

This result is exact for the ZRP in an infinite number of dimensions, and is also expected to hold above an upper critical dimension. Significantly, numerics indicate that this is the correct result in one dimension for asymmetric hopping dynamics. However, as discussed in [67], below the upper critical dimension both dimensionality and bias in the hopping dynamics may be expected to play an important role, and numerics indicate that for symmetric dynamics the exponent changes to $\delta = 1/3$. In order to address these issues further we turn to an alternative, heuristic scaling approach.

4.5.2. Random walk argument. This approach, given in [60], is based on the observation (also observed in the mean-field approach) that the dynamics can be divided into distinct regimes: (i) a nucleation regime, during which the excess $(\rho - \rho_c)L$ particles accumulate at a finite number of condensate sites such that each condensate site contains a number of particles of order L —the remaining sites, which we refer to as bulk sites, have converged to

the steady state with density ρ_c ; (ii) a coarsening regime, during which the condensate sites exchange particles through the bulk—the bulk is viewed as a homogeneous background during this process. The exchange of particles between condensate sites leads to the growth of larger condensates at the expense of smaller ones. This, in turn, leads to a decrease in the number of condensate sites and an increase in the mean condensate size $\langle m(t) \rangle$. Thus the coarsening regime persists until only a single condensate site remains. The random walk argument yields a prediction of the exponent δ appearing in the scaling law (67) which determines the growth of $\langle m(t) \rangle$, the mean condensate size.

We now outline how the random walk argument is applied to the ZRP in one dimension to obtain predictions for the exponent δ for both asymmetric dynamics and symmetric dynamics. During the coarsening regime, the mean rate at which particles hop from bulk sites is given by $\langle u \rangle = 1$. However, at condensate sites particles escape with a rate $u(n) - \langle u \rangle = b/n$, where $n = \mathcal{O}[(\rho - \rho_c)L]$. Therefore a particle escapes from a condensate site in a time that scales as $\mathcal{O}(L)$. We must now find the time it takes for such a particle to reach the next condensate site. This time depends on the symmetry of the hopping dynamics.

- (a) *Asymmetric dynamics.* In this case, the particle reaches the next condensate site to the right with certainty. The particle mobility is given by the mean hop rate from occupied sites in the bulk, which is a finite constant given by $\langle u \rangle / (1 - p(0))$. The average distance between two condensate sites scales as $\mathcal{O}(L)$ therefore the typical time a particle spends in the bulk, having escaped from a condensate site, scales as $\mathcal{O}(L)$. This time is of the same order in L as the typical escape time from condensate sites therefore there are only a finite number of excess particles in the bulk at any time. Thus the transit time between two condensate sites does not limit the dynamics of the coarsening in any way. The growth of the mean condensate size is determined only by the time it takes for a condensate site to lose all of its particles. This time scales as $\mathcal{O}(L \times L)$ therefore the normalized mean condensate size grows like

$$\frac{\langle m(t) \rangle}{(\rho - \rho_c)L} \sim \left(\frac{t}{\tau} \right)^{1/2}, \quad (69)$$

where the time scale for the coarsening regime, $\tau \sim \mathcal{O}(L^2)$. Therefore the prediction $\delta = 1/2$ for asymmetric dynamics is the same as for the mean-field analysis.

- (b) *Symmetric dynamics.* In this case, having left a condensate site, a particle performs a symmetric random walk with a diffusion constant given by $\langle u \rangle / (1 - p(0))$. Now, the probability that a particle reaches the next condensate site, which is determined by the solution of the ‘gamblers’ ruin’ problem [71], is proportional to the inverse separation of condensate sites. Thus this probability is of order L^{-1} , so particles almost certainly return to the condensate site they have just left. The typical time it takes for a particle to escape from a condensate site is therefore $\mathcal{O}(L^2)$. The transit time for such a particle is given by the first passage time for a symmetric random walk which is proportional to the square of the distance, i.e. $\mathcal{O}(L^2)$. Therefore, as in the asymmetric case, this time is of the same order in L as the typical escape time from condensate sites so the growth of $\langle m(t) \rangle$ is determined only by the time it takes a condensate site to lose all of its particles. This time scales as $\mathcal{O}[L \times L^2]$ therefore the normalized mean condensate size grows like

$$\frac{\langle m(t) \rangle}{(\rho - \rho_c)L} \sim \left(\frac{t}{\tau} \right)^{1/3}, \quad (70)$$

where the coarsening time scale $\tau \sim \mathcal{O}(L^3)$. Thus $\delta = 1/3$ for symmetric dynamics.

The results of simulations are consistent with the predictions of the random walk argument in both asymmetric and symmetric cases [67, 60].

The random walk argument can also be applied to the ZRP in more than one dimension [72]. The condensate sites may be viewed as ‘trapping’ sites amongst which the excess particles are exchanged. The key feature of random walks in higher dimensions is that the probability that a particle does not return to the condensate site it has just vacated is $\mathcal{O}(1)$ for all walks except for the symmetric random walk in two dimensions. For symmetric random walks in two dimensions there are logarithmic corrections: the escape probability is $\mathcal{O}[(\ln L)^{-1}]$, where L is now the number of sites in the system. Therefore the upper critical dimension for the coarsening of the ZRP is 2 and above two dimensions, we expect the coarsening to be determined by the mean-field results for the scaling function and exponent, $\delta = 1/2$.

The coarsening time scales can be summarized as follows,

$$\tau \sim \begin{cases} L^3 & \text{in } d = 1, \\ L^2 \ln L & \text{in } d = 2, \\ L^2 & \text{in } d > 2, \end{cases}$$

for symmetric dynamics, and

$$\tau \sim L^2 \quad \forall d, \tag{71}$$

for asymmetric dynamics.

Note that mean-field analysis reproduces the correct exponent for the asymmetric one-dimensional case. The reason for this is that random walk arguments yield the same result, $\delta = 1/2$, whenever the escape probability is of order $\mathcal{O}(1)$, and this is the situation in the one-dimensional case with asymmetric dynamics.

5. Heterogeneous system

5.1. Condensation

We now turn to mechanisms of condensation and the associated coarsening dynamics in the heterogeneous ZRP. In a heterogeneous system the single-site weights $f_l(n)$ depend on the site l . Taking the most general case discussed in section 2.3 (where the hop rate from l to k is $u_l(n_l)W_{kl}$) we have

$$f_l(n) = s_l^n \prod_{i=1}^n \frac{1}{u_l(i)}. \tag{72}$$

Working in the grand canonical ensemble as in section 4.1 one finds

$$\mathcal{Z}_L(z) = \sum_{\{m_l=0\}} z^{\sum_l m_l} \prod_{l=1}^L f_l(m_l) = \prod_l F_l(z) \tag{73}$$

where

$$F_l(z) = \sum_{m=0}^{\infty} z^m f_l(m), \tag{74}$$

and (32) becomes

$$\rho = \frac{z}{L} \sum_l \frac{F'_l(z)}{F_l(z)}. \tag{75}$$

To simplify things we consider a one-dimensional lattice so that $s_l = 1$ and we also take the hop rate of the form $u_l(n) = u_l$, i.e., there is no dependence on the number of particles at the departure site. In this case f_l is given by

$$f_l(n) = \left(\frac{1}{u_l}\right)^n. \quad (76)$$

At this point, the mapping to an ideal Bose gas is evident: the N particles of the ZRP are viewed as bosons which may reside in L states with energies E_l determined by the site hopping rates: $\exp(-\beta E_l) = 1/u_l$. Thus the ground state corresponds to the site with the lowest hopping rate u_{\min} .

We can sum the geometric series (74) to obtain $F_l = 1/(1 - z/u_l)$ and

$$\rho = \frac{z}{L} \sum_l \frac{1}{u_l - z}. \quad (77)$$

Thus the maximum allowed value of z is u_{\min} . In the large L limit we may proceed as in the usual theory of Bose condensation [61] and write

$$\rho = \frac{\langle n_{\min} \rangle}{L} + \int_{u_{\min}}^{\infty} du \mathcal{P}(u) \frac{z}{u - z} \quad (78)$$

where $\langle n_{\min} \rangle$ is the average number of particles at the slowest site and $\mathcal{P}(u)$ is the probability distribution of site hopping rates. Interpreting $\mathcal{P}(u)$ as a density of states, equation (78) corresponds to the condition that in the grand canonical ensemble of an ideal Bose gas the number of bosons per state is ρ . If the density of states vanishes as $u \searrow u_{\min}$ as

$$\mathcal{P}(u) \sim (u - u_{\min})^\gamma \quad \gamma > 0 \quad (79)$$

then the second term in (78) will be finite as $z \nearrow u_{\min}$ and will define ρ_c . For $\rho > \rho_c$, the first term in (78) must be finite thus $\langle n_{\min} \rangle \sim O(L)$. This is the mechanism of condensation underlying heterogeneous models of networks, for example [11].

5.2. Single defect site

A very simple example, which serves to illustrate the mechanism of condensation, is to have just one ‘slow site’, i.e., $u_1 = p < 1$ while the other $L - 1$ sites have hopping rates $u_l = 1$ when $l > 1$ [6]. The simplicity of this system actually allows the canonical partition function (3) to be calculated. This is easy to write down as

$$Z_{L,N} = \sum_{n=0}^N p^{-n} \sum_{n_2, \dots, n_L} \delta \left(\sum_{\mu=2}^L n_\mu - N + n \right) \quad (80)$$

where n counts the number of particles on the slow site. Identifying the sum over n_2, \dots, n_L as the number of ways of distributing $N - n$ indistinguishable balls in $L - 1$ boxes implies that

$$Z_{L,N} = \sum_{n=0}^N \binom{N+L-n-2}{L-2} p^{-n}. \quad (81)$$

In order to determine which terms dominate the sum (81) we look for the stationary point of the summand, which occurs when the ratio of consecutive terms $n, n - 1$ is equal to 1:

$$\frac{1}{p} \frac{N - n + 1}{(N + L - n - 1)} = 1. \quad (82)$$

For L large, we find a solution $n \simeq L(\rho - \rho_c)$ when $\rho > \rho_c = p/(1 - p)$. For $\rho < \rho_c$, however, we do not find a solution for positive n therefore the maximum is the boundary term $n = 0$ and the sum will be dominated by terms $n \sim \mathcal{O}(1)$. Thus $\rho_c = p/(1 - p)$ is the critical density above which we have a condensate: we see clearly that in the condensed phase the defect site serves to absorb the excess number of particles $L(\rho - \rho_c)$; in the fluid phase the number of particles at the defect site is $\mathcal{O}(1)$.

In the fluid phase, we evaluate the asymptotic behaviour of the partition function by expanding for small n , since these terms dominate the sum:

$$Z_{L,N} \simeq \sum_{n=0}^N \binom{N+L}{L} \frac{L^2 N^n}{(N+L)^{n+2}} p^{-n} \simeq \binom{N+L}{L} \frac{\rho_c}{(1+\rho)(\rho_c - \rho)}. \tag{83}$$

In the condensed phase, one approximates the sum by an integral and uses Stirling's formula to obtain

$$Z_{L,N} \simeq L^{1/2} \int_0^\rho \frac{dx}{\sqrt{2\pi}} \left(\frac{1+\rho-x}{\rho-x} \right)^{1/2} \exp L[(1+\rho-x) \ln(1+\rho-x) - (\rho-x) \ln(\rho-x) - x \ln p] \tag{84}$$

where $x = n/L$. Then the integral is dominated by $x \simeq \rho - \rho_c$ and expanding to second order and performing the Gaussian integral yields

$$Z_{L,N} \simeq (1-p)^{-(L+1)} p^{-N}. \tag{85}$$

From expressions (83), (85) and (11), the mean hop rate in the two phases is given by

$$\langle u \rangle = \frac{\rho}{1+\rho} \quad \rho < p/(1-p), \tag{86}$$

$$\langle u \rangle = p \quad \text{for } \rho > p/(1-p). \tag{87}$$

These expressions for $\langle u \rangle$ are easily understood as the hop rate being limited by the condensate in the condensed phase and by the probability of empty sites, $p(0) = 1/(1+\rho)$, in the fluid phase.

5.2.1. Defect with n-dependent hop rate. Recently, the case of a single defect site with an occupation number dependent hop rate $u_1(n)$ has been considered. This allows one to investigate the interplay between condensation driven by particle number dependent hop rates and condensation driven by heterogeneity [65].

Again one can write down the canonical partition function. Taking

$$u_1(n) = p \left(1 + \frac{b}{n} \right) \quad \text{for } n > 0$$

with $p < 1$ and $u_l(n) = 1$ for $l \neq 1$ yields

$$Z_{L,N} = \sum_{n=0}^N p^{-n} \frac{n!}{(1+b)_n} \binom{N+L-n-2}{L-2}, \tag{88}$$

where $(1+b)_n$ is a Pochhammer symbol defined in (57). One can again determine the stationary point of the summand by setting the ratio of consecutive terms equal to 1, resulting in a quadratic expression

$$\frac{n}{p(b+n)} \frac{(N-n+1)}{(N+L-n-1)} = 1. \tag{89}$$

A real, non-negative solution of this quadratic exists for

$$\rho > \rho_2 = \frac{p}{1-p} + \frac{2}{1-p} \left(\frac{bp}{L} \right)^{1/2} + \mathcal{O}(1/L). \quad (90)$$

As $L \rightarrow \infty$ one has a condensation transition at $\rho_c = p/(1-p)$, which is independent of b . Thus the condensation is triggered by the site being slow rather than the n -dependence of $u_1(n)$. However, on any large but finite system (with $b > 0$) the condensed phase does not appear until $\rho_2 = \rho_c + \mathcal{O}(L^{-1/2})$. Thus on a finite system the fluid phase continues above the thermodynamic critical density. This also results in the mean hop rate going above its predicted thermodynamic value. At $\rho = \rho_2$ one sees the system wander between a fluid state and a condensed state [65].

5.3. Dynamics

The coarsening behaviour of the heterogeneous ZRP can be obtained by using a scaling argument to predict the growth law for the typical condensate size. We consider the case where the hop rates are drawn from the distribution with support $u \in [u_{\min}, 1]$

$$\mathcal{P}(u) = \frac{1+\gamma}{(1-u_{\min})^{\gamma+1}} (u-u_{\min})^\gamma, \quad \gamma > 0. \quad (91)$$

The argument is rather similar to the homogeneous case. After an initial nucleation regime, most of the system relaxes to the steady state except for a finite number of sites which contain the excess particles in the system. Eventually, only two condensate sites will remain and these will be the sites with the two slowest hop rates. These two sites will be separated by a distance of order $\mathcal{O}(L)$ (if they were separated by a distance of order $\mathcal{O}(1)$ they would effectively act as a single slow site). This implies that the second slowest site must contain a number of particles of order $\mathcal{O}(L)$, since it is the slowest site encountered by particles in a finite fraction of the system. This is to be compared with the steady state, in which the j th slowest site contains a number of particles of order $\mathcal{O}((L/j)^{1/(1+\gamma)})$ which is subextensive for $\gamma > 0$. This scaling can be inferred from an extremal statistics argument.

In the case of asymmetric dynamics [73–75], the second slowest site gains particles from the left with rate u_{\min} , but loses particles to the right with rate $u_1 < u_{\min}$, thus it loses particles with a net rate $\Delta u = u_1 - u_{\min} \sim \mathcal{O}(L^{-1/(1+\gamma)})$. Therefore the time τ this site takes to lose all its excess particles (i.e., excess above its typical steady state occupancy) scales as $\mathcal{O}(L/\Delta u)$. Thus

$$\tau \sim L^{(2+\gamma)/(1+\gamma)}, \quad (92)$$

and the normalized mean condensate size grows like

$$\frac{\langle m(t) \rangle}{(\rho - \rho_c)L} \sim \left(\frac{t}{\tau} \right)^{\frac{1+\gamma}{2+\gamma}}, \quad (93)$$

so the exponent $\delta = (1+\gamma)/(2+\gamma)$ for asymmetric dynamics.

For symmetric dynamics a similar argument can be used except that now the coarsening time scale τ is augmented by a factor L due to the probability that a particle returns to the condensate site it has just vacated rather than reaching the slowest site [76]. Thus

$$\tau \sim L^{(3+2\gamma)/(1+\gamma)}, \quad (94)$$

and the normalized mean condensate size grows like

$$\frac{\langle m(t) \rangle}{(\rho - \rho_c)L} \sim \left(\frac{t}{\tau} \right)^{\frac{1+\gamma}{3+2\gamma}}, \quad (95)$$

so the exponent $\delta = (1+\gamma)/(3+2\gamma)$ for symmetric dynamics. This was shown for $\gamma > 1$ in [76]; however it in fact holds for $\gamma > 0$ [77].

6. Generalizations

6.1. Two-species ZRP

In view of the recent interest in two component models [78] it is natural to generalize the ZRP to a model with two species of conserved particles. In this way, one can explore the role of conservation laws in the ZRP. In particular, considering the generic nature of the condensation mechanism in the single-species ZRP, it is of interest to look for other mechanisms of condensate formation which may have some generic applicability. Moreover, as discussed in section 3, the behaviour of shaken granular gases in which the grains come in one of two sizes (i.e., a bidisperse system), and the behaviour of networks with directed edges, can be understood in terms of a two-species ZRP. The particle dynamics we consider are chosen such that the evolution of the two particle species are coupled, and the nature of the coupling can be chosen such that condensation of one of the species is induced by the other. Such an interplay arises in models of particles moving on an evolving disordered background [79–81]. We discuss how this interplay leads to new mechanisms of condensation and rich coarsening behaviour.

In one dimension, we define the two-species ZRP on a lattice containing L sites and with periodic boundary conditions. At each site l , there are n_l particles of species A and m_l particles of species B . The total number of species A particles in the system is N and the total number of species B particles is M . These particles hop from site l to the neighbouring site to the right, species A with rate $u(n_l, m_l)$ and species B with rate $v(n_l, m_l)$.

6.1.1. Steady state factorization condition. Happily, one of the most useful features of the single-species model—the factorized steady state—remains a feature of the two-species model provided the dynamics satisfy a certain constraint [82, 83]. That is, one can still express the probabilities $P(\{n_l\}; \{m_l\})$ in the factorized form

$$P(\{n_l\}; \{m_l\}) = Z_{L,N,M}^{-1} \prod_{l=1}^L f(n_l, m_l), \quad (96)$$

provided the hop rates satisfy the constraint

$$\frac{u(n_l, m_l)}{u(n_l, m_l - 1)} = \frac{v(n_l, m_l)}{v(n_l - 1, m_l)}, \quad (97)$$

for $n_l, m_l \neq 0$. The choices of $u(n_l, 0)$ and $v(0, m_l)$ remain unconstrained. When the hop rates satisfy (97), we can find single-site weights $f(n, m)$ given by

$$f(n, m) = \prod_{i=1}^n [u(i, m)]^{-1} \prod_{j=1}^m [v(0, j)]^{-1}, \quad (98)$$

and with $f(0, 0) = 1$. We remark that the hop rates do indeed play a symmetric role in (98), but that this symmetry is obscured within the constraint equation (97). Finally, the normalization $Z_{L,N,M}$, equivalent to the canonical partition function, is given by

$$Z_{L,N,M} = \sum_{\{n_l\}, \{m_l\}} \prod_{l=1}^L f(n_l, m_l) \delta\left(\sum_{l=1}^L n_l - N\right) \delta\left(\sum_{l=1}^L m_l - M\right), \quad (99)$$

where the δ -functions ensure that we only sum over configurations with N particles of species A and M of species B .

The steady state (96) to (99) for this model can be derived in much the same way as in the single-species case. One begins by writing down the steady state condition on the probabilities $P(\{n_l\}; \{m_l\})$. This has the form

$$0 = \sum_{l=1}^L [\{u(n_{l-1} + 1, m_{l-1})P(\dots, n_{l-1} + 1, n_l - 1, \dots; \dots, m_{l-1}, m_l, \dots) - u(n_l, m_l)P(\dots, n_{l-1}, n_l, \dots; \dots, m_{l-1}, m_l, \dots)\}\theta(n_l) + \{v(n_{l-1}, m_{l-1} + 1)P(\dots, n_{l-1}, n_l, \dots; \dots, m_{l-1} + 1, m_l - 1, \dots) - v(n_l, m_l)P(\dots, n_{l-1}, n_l, \dots; \dots, m_{l-1}, m_l, \dots)\}\theta(m_l)] \quad (100)$$

where the Heaviside function $\theta(x_l)$ ensures that site l is occupied in order for a particle either to have arrived or to be able to vacate there. Now, the first term on the rhs of (100) is a gain term due to an A particle hopping into site l from site $l - 1$ and the second term is a loss term due to an A particle hopping out of site l ; the third and fourth terms represent analogous processes for the B particles. As in the single-species case, we simply insert the factorized form (96) into (100), but now we ask that the gain and loss terms due to the dynamics of the A particles cancel independently of the gain and loss terms due to the dynamics of the B particles. We look to achieve this cancellation for each term l in the sum separately hence

$$u(n_l, m_l) f(n_{l-1}, m_{l-1}) f(n_l, m_l) = u(n_{l-1} + 1, m_{l-1}) f(n_{l-1} + 1, m_{l-1}) f(n_l - 1, m_l), \quad (101)$$

for all $n_l \neq 0$, and

$$v(n_l, m_l) f(n_{l-1}, m_{l-1}) f(n_l, m_l) = v(n_{l-1}, m_{l-1} + 1) f(n_{l-1}, m_{l-1} + 1) f(n_l, m_l - 1), \quad (102)$$

for all $m_l \neq 0$, for all values of l and after cancelling common factors. These equations in turn imply that

$$\frac{u(n_l, m_l) f(n_l, m_l)}{f(n_l - 1, m_l)} = \frac{u(n_{l-1} + 1, m_{l-1}) f(n_{l-1} + 1, m_{l-1})}{f(n_{l-1}, m_{l-1})} = \text{constant}, \quad (103)$$

$$\frac{v(n_l, m_l) f(n_l, m_l)}{f(n_l, m_l - 1)} = \frac{v(n_{l-1}, m_{l-1} + 1) f(n_{l-1}, m_{l-1} + 1)}{f(n_{l-1}, m_{l-1})} = \text{constant}, \quad (104)$$

for all l . Both constants are set equal to unity without loss of generality, since they only appear as overall factors in the normalization. The two relations (103) and (104) are iterated to yield (98). However, (103) and (104) also imply the constraint (97) on the choice of $u(n_l, m_l)$ and $v(n_l, m_l)$. To obtain this constraint, we use (103) and (104) to obtain two expressions for $f(n_l, m_l)$ in terms of $f(n_l - 1, m_l - 1)$:

$$f(n_l, m_l) = \frac{f(n_l - 1, m_l - 1)}{u(n_l, m_l) v(n_l - 1, m_l)} = \frac{f(n_l - 1, m_l - 1)}{u(n_l, m_l - 1) v(n_l, m_l)}. \quad (105)$$

Since both of these expressions must give the same result the hop rates are required to obey the constraint (97). Equation (105) also indicates how we might interpret physically the constraint (97): it suggests that the relationship between $f(n_l - 1, m_l - 1)$ and $f(n_l, m_l)$ is the same when an A particle hops from l and then a B , as when the B particle hops before the A . In other words, the single-site weight $f(n_l, m_l)$ is independent of the order in which the particle species arrived at l .

As in the single-species case, it is straightforward to generalize the derivation of the steady state to an arbitrary lattice, including generalizations to disorder in the hop rates, partial asymmetry in the dynamics and any dimension. It is also possible to generalize to any number of species, Q say, in which case each species satisfies $Q - 1$ constraints of the form (97), one for every pair of species [82].

6.1.2. *Condensation.* We now consider how the interaction of the two species may lead to a condensation transition as a function of the particle densities,

$$\rho_A = \frac{N}{L} \quad \text{and} \quad \rho_B = \frac{M}{L}, \quad (106)$$

for species A and B respectively. We consider a case where the dynamics of one of the particle species—the B particles—depends only on the number of particles of the other species at the departure site. Hence we take

$$v(n, m) = 1 + \frac{c}{(n+1)^\gamma}, \quad (107)$$

for all n and for $m > 0$, where $c, \gamma > 0$. This choice for $v(n, m)$ now determines the m -dependence we must take for $u(n, m)$ through the constraint (97):

$$u(n, m) = \left(\frac{1 + \frac{c}{(n+1)^\gamma}}{1 + \frac{c}{n^\gamma}} \right)^m. \quad (108)$$

We could choose to multiply $u(n, m)$ by some function of n and still satisfy (97) but we will proceed with just the rate (108) for simplicity. The single-site weights $f(n, m)$, determined by (98), are given by

$$f(n, m) = \left(1 + \frac{c}{(n+1)^\gamma} \right)^{-m}. \quad (109)$$

As in the single-species case, it is simplest to work within the grand canonical ensemble in order to demonstrate condensation. The grand canonical partition function is defined as

$$\mathcal{Z}_L(z, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z^n y^m Z_{L,n,m}, \quad (110)$$

where we now have two fugacities, z and y , which fix the particle densities through

$$\rho_A = \frac{z}{L} \frac{\partial \ln \mathcal{Z}_L(z, y)}{\partial z}, \quad \rho_B = \frac{y}{L} \frac{\partial \ln \mathcal{Z}_L(z, y)}{\partial y}. \quad (111)$$

Now, we use (99) and perform the sums over n and m in (110) to obtain

$$\mathcal{Z}_L(z, y) = \sum_{\{n_l\}, \{m_l\}} z^{\sum_l n_l} y^{\sum_l m_l} \prod_{l=1}^L f(n_l, m_l) = [F(z, y)]^L, \quad (112)$$

where

$$F(z, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z^n y^m f(n, m), \quad (113)$$

and the densities are determined by

$$\rho_A = z \frac{\partial \ln F(z, y)}{\partial z}, \quad \rho_B = y \frac{\partial \ln F(z, y)}{\partial y}. \quad (114)$$

In the single-species case, the critical density was determined by the properties of $F(z)$ and its derivative when z assumed its maximum value β . A similar picture emerges here, which we illustrate for $f(n, m)$ given by (109).

Using (109) in (113), and performing the sum over m , one obtains

$$F(z, y) = \sum_{n=0}^{\infty} z^n \frac{(1+n)^\gamma + c}{(1+n)^\gamma (1-y) + c}, \quad (115)$$

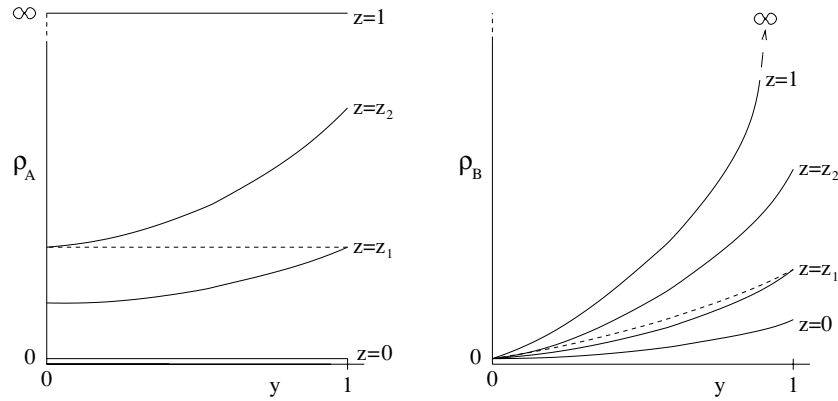


Figure 6. The solid lines represent schematic dependences of the particle densities ρ_A and ρ_B for fixed values of z and as a function of y . The dashed line in the right-hand graph illustrates how ρ_B varies as a function of z and y given that ρ_A is fixed.

$$z \frac{\partial}{\partial z} F(z, y) = \sum_{n=0}^{\infty} n z^n \frac{(1+n)^\gamma + c}{(1+n)^\gamma (1-y) + c}, \quad (116)$$

$$y \frac{\partial}{\partial y} F(z, y) = \sum_{n=0}^{\infty} (1+n)^\gamma z^n \frac{y((1+n)^\gamma + c)}{[(1+n)^\gamma (1-y) + c]^2}. \quad (117)$$

The radii of convergence of these series are $z = 1$ and $y = 1$. We can now use equations (115) to (117) in (114) to find z and y , given values for ρ_A and ρ_B . In figure 6, we illustrate how ρ_A and ρ_B depend on y for fixed z . This figure is obtained by noting that for a given z , both ρ_A and ρ_B are monotonically increasing functions of y . Then we can infer the behaviour shown in figure 6 by analysing (114) in the four limits $y \rightarrow 0$, $z \rightarrow 0$, $y \rightarrow 1$ and $z \rightarrow 1$. The existence of a condensation transition is demonstrated by considering a system containing a density ρ_A of A particles, represented by the dashed lines in figure 6. The solution of (114) requires that z and y lie in the range $z_1 \leq z \leq z_2$ and $0 \leq y \leq 1$. In this range, ρ_B increases monotonically from $\rho_B = 0$, where $z = z_2$ and $y = 0$, to its maximum value, where $y = 1$ and $z = z_1$. Thus, for every finite value of ρ_A there exists a finite maximum value of ρ_B above which we can no longer solve (114). When ρ_B exceeds this maximum (114) cannot be satisfied and a condensation transition ensues. Hence we obtain the phase diagram shown in figure 7. For $\gamma = 1$ the critical line is given simply by

$$\rho_B = (1 + \rho_A)/c. \quad (118)$$

Thus the model exhibits a phase transition between a fluid phase, in which both particle species are distributed exponentially, and a condensed phase, in which ρ_B exceeds a maximum critical value and the excess B particles condense onto a single site. The number of B particles in this condensate is proportional to the system size L —in the limit $N, M, L \rightarrow \infty$ it contains a finite fraction of an infinite number of particles. However, we note that for $m \rightarrow \infty$, $u(n, m) \rightarrow 0$ if n is finite. But this is not consistent with continuity since the current of A particles away from the condensate site is finite therefore $u(n, m)$ at the condensate site must also be finite. Thus it must be that $n \rightarrow \infty$ at the condensate in order that the current out of the condensate site is finite. For large n and m , $u(n, m) \sim \exp(-\gamma m/n^{1+\gamma})$ and since m is a number of order $m \sim \mathcal{O}(L)$ at the condensate site, n must be of order $n \sim \mathcal{O}(L^{1/(1+\gamma)})$ for

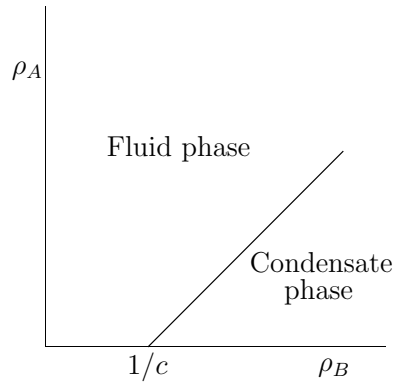


Figure 7. Phase diagram for the two-species zero-range process with $\gamma = 1$.

finite $u(n, m)$. Thus the condensate of B particles is sustained by a subextensive condensate of A particles.

The conditions on $u(n, m)$ and $v(n, m)$, and therefore $f(n, m)$ also, leading to condensation have been considered for general rates in [84]. Additional condensation transitions can be obtained by supplementing the A particle hop rate (108) with a factor $(1 + b/n)$. In this case two additional condensed phases arise, one in which only the A particles condense, and one in which both particle species condense—thus this phase also exhibits induced condensation of one species by the other.

6.1.3. Dynamics of condensation. The coarsening dynamics in the condensed phase shown in figure 7 can be obtained by adapting the random walk arguments given for the single-species case [85].

Thus, as before, we consider a system most of which has relaxed to the steady state, except for a finite number of condensate sites. The condensate sites contain a number of A particles of order $n \sim \mathcal{O}(L^{1/(1+\gamma)})$ and a number of B particles of order $m \sim \mathcal{O}(L)$, i.e. $m \sim n^{1+\gamma}$. Therefore, from (107), the B particle hop rate decays to its steady state value as $c/m^{\gamma/(1+\gamma)}$, so the typical time scale over which B particles escape from condensate sites scales as $L^{\gamma/(1+\gamma)}$. Then the time it takes for a condensate site to lose all of its B particles scales as $L^{1+\gamma/(1+\gamma)}$. This sets the coarsening time scale, thus the growth of the mean B particle condensate size follows the scaling law (67) with exponent

$$\delta = \frac{1 + \gamma}{1 + 2\gamma}. \tag{119}$$

A prediction for the exponent δ can also be obtained in the case that the dynamics are symmetric. The dynamics are altered in the same way as in the single-species case, in that only every L th particle on average reaches the next condensate site rather than returning to the one it has just departed, thus the typical time scale over which B particles escape from condensate sites is a factor of L longer: it scales as $L^{1+\gamma/(1+\gamma)}$. Therefore the time it takes for a condensate site to lose all of its B particles scales as $L^{2+\gamma/(1+\gamma)}$ and the coarsening exponent is given by

$$\delta = \frac{1 + \gamma}{2 + 3\gamma}. \tag{120}$$

It is interesting to note that these exponents are the same as those obtained for the heterogeneous ZRP with hop rates drawn from the distribution (cf section 5.3)

$$\mathcal{P}(u) = [(\gamma^{-1} + 1)/(1 - u_{\min})^{\gamma^{-1}+1}](u - u_{\min})^{\gamma^{-1}}. \tag{121}$$

To consider why this might be the case, one views the dynamics defined in (107), (108) as a model of particles (the B particles) moving on an evolving disordered background (given by the A particles). By the time the coarsening regime has been reached, at the condensate sites the evolving disorder is effectively quenched. Therefore it is not necessarily surprising that the two models exhibit similar coarsening behaviour for some distribution $\mathcal{P}(u)$. The reason the form (121) is the relevant one for the rates (107), (108) is as follows. In the heterogeneous system, as discussed in section 5.3, the coarsening is governed by the difference between the hop rates at the two slowest sites in the system, Δu . For the distribution (121), $\Delta u \sim L^{-1/(1+\nu^{-1})}$. This rate separation plays the role of the background of A particles in the two-species model. The remaining contributions to the coarsening time scale in the heterogeneous model are then determined by the symmetry of the hopping dynamics, i.e. the coarsening time scale is given by a factor of order L for asymmetric dynamics, or a factor of order L^2 for symmetric dynamics, multiplied by the inverse rate separation. This leads to the same exponents as those obtained in the condensed phase of the two-species model.

6.2. Urn models and the Misanthrope process

A natural extension of the ZRP dynamics is to hop rates which depend not only on the occupancy of the departure site, but also on that of the target site. For example, a recent application of such models is to rewiring dynamics in networks as we discuss in section 6.2.4. Models of this kind are often referred to as ‘urn models’. As reviewed in [86] urn models comprise balls distributed amongst a number of boxes with conserving dynamics for the exchange of balls. These dynamics are usually defined on a fully connected geometry and obey detailed balance with respect to some energy function defined as a sum of single-site energies.

On the other hand the Misanthrope process, introduced in the mathematical literature in [87], is a model with dynamics defined without reference to an energy function. In one dimension its dynamics are defined in exactly the same way as the ZRP except that now the hop rate depends on the occupancy of both departure and target sites. Thus a particle hops from site l to site $l + 1$ with a rate $u(n_l, n_{l+1})$.

6.2.1. Steady state factorization condition. It turns out that the steady state still has a factorized form, provided the hop rates satisfy a constraint. To see this, consider the condition on the steady state probabilities $P(\{n_l\})$:

$$0 = \sum_{l=1}^L [u(n_{l-1} + 1, n_l - 1)P(\dots, n_{l-1} + 1, n_l - 1, \dots)\theta(n_l) - u(n_{l-1}, n_l)P(\{n_l\})\theta(n_{l-1})]. \quad (122)$$

To look for a factorized steady state of the form (2) we introduce a counterterm which cancels under the sum and then look to cancel each term in the sum separately, hence

$$\begin{aligned} \bar{f}(n_{l-1})f(n_l) - f(n_{l-1})\bar{f}(n_l) &= u(n_{l-1} + 1, n_l - 1)f(n_{l-1} + 1)f(n_l - 1)\theta(n_l) \\ &\quad - u(n_{l-1}, n_l)f(n_{l-1})f(n_l)\theta(n_{l-1}), \end{aligned} \quad (123)$$

having cancelled common factors (a product over the functions $f(n_k)$ at all sites $k \neq l - 1, l$), where $\bar{f}(n)$ is some function to be determined. The lhs of (123) represents the counterterm. We remark that this cancellation mechanism is identical to that of the so-called ‘matrix product ansatz’ [53] in the case where matrices are replaced by the functions $f(n)$ and $\bar{f}(n)$ [88].

In order to satisfy (123) for all values of n_{l-1} and n_l there are now three cases to consider:

- $n_{l-1} = 0, n_l \neq 0$. In this case

$$\bar{f}(0)f(n_l) - f(0)\bar{f}(n_l) = u(1, n_l - 1)f(1)f(n_l - 1). \quad (124)$$

- $n_{l-1} \neq 0, n_l = 0$. In this case

$$\bar{f}(n_{l-1})f(0) - f(n_{l-1})\bar{f}(0) = -u(n_{l-1}, 0)f(n_{l-1})f(0). \quad (125)$$

The lhs of (124) and (125) are both of the same form, therefore we are able to eliminate $\bar{f}(n)$ from these two equations which yields

$$f(n) = \frac{u(1, n - 1)}{u(n, 0)} \frac{f(1)}{f(0)} f(n - 1), \quad (126)$$

for $n > 0$. This recursion is easily iterated to obtain the factors $f(n)$ in terms of the hop rates:

$$f(n) = f(0) \left(\frac{f(1)}{f(0)} \right)^n \prod_{i=1}^n \left(\frac{u(1, i - 1)}{u(i, 0)} \right). \quad (127)$$

We can now substitute (126) into (124) or (125) to find the expression for $\bar{f}(n)$, hence

$$\bar{f}(n) = f(n)[\bar{f}(0) - u(n, 0)]. \quad (128)$$

- $n_{l-1} \neq 0, n_l \neq 0$. We have used the first two cases to derive expressions for $f(n)$ and $\bar{f}(n)$. Thus we must now satisfy (123) under substitution of (126) and (128). We find that (123) is indeed satisfied provided the hop rates satisfy the condition

$$u(n, m) - u(n + 1, m - 1) \frac{u(1, n)u(m, 0)}{u(n + 1, 0)u(1, m - 1)} = u(n, 0) - u(m, 0). \quad (129)$$

This condition reduces to two separate conditions

$$u(m, n) = u(n + 1, m - 1) \frac{u(1, n)u(m, 0)}{u(n + 1, 0)u(1, m - 1)} \quad (130)$$

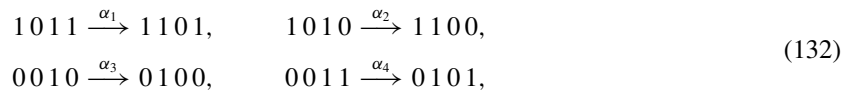
$$u(n, m) - u(m, n) = u(n, 0) - u(m, 0) \quad (131)$$

for all values of n and for $m > 0$.

Hence the steady state of the Misanthrope process is given by the factorized form (2) with $f(n)$ given by (127) provided the hop rates satisfy the constraint (129). We remark that if we take $u(n, m)$ to be a function of n , the number of particles at the departure site only, we recover the ZRP, albeit via a slightly different derivation to the one presented in section 2.

6.2.2. Mapping to the KLS model. Just as for the ZRP, the Misanthrope process can be mapped onto an exclusion process. This mapping is illustrated in figure 8. It is achieved in the same way as for the ZRP—sites in the Misanthrope process become occupied sites in the exclusion process and particles in the Misanthrope process become vacant sites in the exclusion process—the only difference is that now the hop rates depend not only on the distance to the next particle in front, but also on the distance to the previous particle behind.

An example of this mapping is between the Misanthrope process and the KLS model [89]. The KLS model is an exclusion process in which particles hop from site i to site $i - 1$ with a rate depending on the occupancy of sites $i - 2$ and $i + 1$. The dynamics are defined by the processes



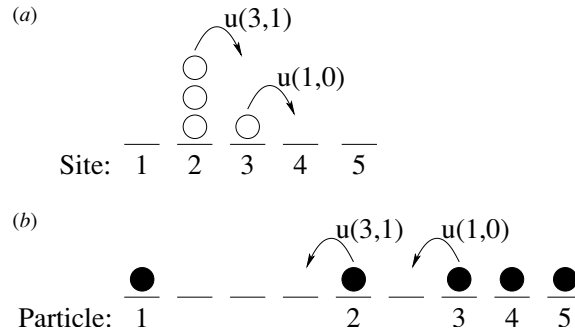


Figure 8. Mapping between the Misanthrope process and the asymmetric exclusion process.

where we use a ‘1’ to represent a particle and a ‘0’ to represent a vacancy. The steady state probabilities $P(\{\eta_i\})$, where the occupation variables $\eta_i = 0$ or 1, can be written in the form

$$P(\{\eta_i\}) = \frac{1}{Z_{L,N}^{(\text{KLS})}} \exp\left(\beta \sum_{i=1}^{L+N} \eta_i \eta_{i+1}\right), \quad (133)$$

for a system with L particles and N vacancies, where $Z_{L,N}^{(\text{KLS})}$ is a normalization. This steady state holds provided [90]

$$\alpha_2 = e^\beta \alpha_4, \quad \alpha_2 + \alpha_4 = \alpha_1 + \alpha_3. \quad (134)$$

The mapping to the Misanthrope process is achieved by making the identification: $u(n, m) = \alpha_3$ for $n > 1, m > 0$; $u(n, 0) = \alpha_4$ for $n > 1$; $u(1, m) = \alpha_2$ for $m > 0$; $u(1, 0) = \alpha_1$. With this choice of $u(n, m)$, condition (129) reduces to (134). This leads to a Misanthrope process for N particles on L sites with the steady state (2) where $f(n)$ is given by

$$f(n) = \left(\frac{f(1)}{f(0)}\right)^n f(0) e^{\beta(n-1)} \quad \text{for } n \geq 1, \quad (135)$$

and where substitution of the Misanthrope rates into (129) yields the second of the conditions in (134). Thus, under the mapping the Ising-like form of the steady state (133) changes into a simple factorized form.

6.2.3. Misanthrope process on fully connected geometry. In this subsection, we consider a model where a particle can hop to any other site with a rate which depends on the occupations of the departure and destination sites. Thus we are considering a Misanthrope process on a fully connected geometry.

Since the sites are fully connected the steady state weight should be invariant under all permutations of the sites. Note, however, that on a finite system this does not imply a factorized steady state. This is perhaps counterintuitive as one might expect the fully connected geometry to yield ‘mean-field’ results and the steady state to factorize. On the other hand, in the limit of infinite system size mean-field results should be recovered and it would be interesting to demonstrate that this is the case.

On a finite system, it turns out that the steady state has a factorized form only if the hop rates satisfy a constraint. This constraint is less restrictive than the asymmetric one-dimensional case discussed above. When this constraint is satisfied detailed balance is obeyed. Thus, in this case the Misanthrope process is equivalent to the urn models reviewed in [86].

To see this, consider the condition on the steady state probabilities,

$$0 = \sum_{l \neq k} [u(n_l + 1, n_k - 1)P(\dots, n_l + 1, \dots, n_k - 1, \dots) - u(n_l, n_k)P(\{n_l\}) + u(n_k + 1, n_l - 1)P(\dots, n_l - 1, \dots, n_k + 1, \dots) - u(n_k, n_l)P(\{n_l\})]. \quad (136)$$

where the sum runs over all pairs of sites l, k . The assumption of a factorized steady state yields

$$0 = \sum_{l \neq k} \left[u(n_l + 1, n_k - 1) \frac{f(n_l + 1)f(n_k - 1)}{f(n_l)f(n_k)} - u(n_l, n_k) + u(n_k + 1, n_l - 1) \frac{f(n_k + 1)f(n_l - 1)}{f(n_k)f(n_l)} - u(n_k, n_l) \right]. \quad (137)$$

It turns out that to satisfy this one must have detailed balance

$$u(n_l + 1, n_k - 1) f(n_l + 1) f(n_k - 1) = u(n_k, n_l) f(n_l) f(n_k) \quad (138)$$

which yields

$$f(n) = f(0) \left(\frac{f(1)}{f(0)} \right)^n \prod_{m=1}^n \frac{u(1, m - 1)}{u(m, 0)} \quad (139)$$

provided the following condition holds for $n > 1, m > 0$

$$u(n, m) = \frac{u(n, 0)u(1, m)}{u(1, n - 1)u(m + 1, 0)} u(m + 1, n - 1). \quad (140)$$

which is one of the two conditions (130) required for the one-dimensional asymmetric case to factorize. A simple way to guarantee the equality of (140) is for $u(n, m)$ to factorize into a function of n and a function of m . This is the case of interest for rewiring networks discussed in the next subsection.

6.2.4. Mapping to network dynamics. Recall that a network is a collection of nodes connected by edges with dynamics for rewiring of edges (see section 3.2). Here we discuss how these dynamics correspond to Misanthrope dynamics.

Let us fix the number of nodes in the network to be L and the number of (undirected) edges to be N . A general rewiring dynamics is as follows [39]: an edge is chosen at random and one end of the edge initially connected to node k is rewired to another node k' with rate $\phi(k')h(k)$. An increasing function $\phi(k')$ describes preferential attachment whereas a decreasing function $h(k)$ describes enhanced detachment of an edge from nodes with low degree. (In the network literature $f(k)$ is usually used to denote $\phi(k)$ but we have chosen to use $\phi(k)$ to avoid a clash of notation with the single-site weights $f(n)$.) Thus the overall rate of a rewiring event such that two nodes with degrees k, k' end up with degrees $k - 1, k' + 1$ is proportional to $k\phi(k')h(k)$. Note that the factor k stems from choosing an edge at random. The dynamics clearly preserves the number of edges and thus generates the ‘canonical ensemble’. Other ensembles of graphs are discussed in [42, 43]. Note, however, that the dynamics may generate self-connected nodes (tadpoles) and multiple edges between two nodes (melons). It has been argued that these graphs have little effect on the resulting ensemble [39].

We now think of the number of edges connected to a node as the number of particles at a site in the Misanthrope process. The rate at which a particle hops from a site with k' to a site with k particles is then $u(k, k') = kh(k)\phi(k')$. Also note that an edge may be rewired from

one node to any other, thus we are considering Misanthrope dynamics on a fully connected geometry. We may therefore invoke the results of 6.2.3 which tell us that if $u(k, k')$ factorizes, which is the case here, we have detailed balance and the steady state factorizes. The single-site weight becomes

$$f(k) = f(0) \left(\frac{f(1)h(1)}{f(0)\phi(0)} \right)^k \prod_{m=1}^k \frac{\phi(m-1)}{mh(m)}. \quad (141)$$

Recalling the results of section 4.2 we see that for condensation to occur we require that $mh(m)/\phi(m-1)$ decay more slowly than $1 + 2/m$. Setting $h(k) = 1$ gives the preferential attachment case. Thus the preferential attachment function $\phi(m)/m$ should increase more quickly than $1 - 1/m$ to have condensation.

An alternative mapping from the dynamics of a network to a ZRP is proposed in [91].

6.2.5. Relation to Backgammon model. The Backgammon model, introduced in [92], is an urn model which can be related to the Misanthrope process. There has been considerable interest in the Backgammon model as a simple model of glassy dynamics. Defined on a fully connected geometry, the idea is that particles hop between sites and there is an energy cost associated with occupied sites. The hopping rate from site l to k is therefore a product of factors for each site and the existence of an energy function guarantees that condition (130) is satisfied. Since there is an energy function, the dynamics can be parametrized by a temperature and at very low temperatures the steady state will be dominated by configurations with very few occupied sites. Glassy dynamics are exhibited in the relaxation towards the steady state, where the elimination of occupied boxes becomes slower and slower.

A simple example of Misanthrope dynamics corresponding to the Backgammon model is $u(n, m) = 1 + (e^{-\beta} - 1)\delta_{m,0}$ for $n > 0$, where β is the inverse temperature.

6.3. General mass transfer model

Given that the steady state of the ZRP factorizes it is natural to ask under what conditions may a factorized steady state be admitted. Recent work [93, 94] has answered this question and revealed an appealingly simple condition for factorization within a framework that encompasses a wide range of models.

The work [93, 94] generalizes the ZRP in three ways:

- the mass is a continuous variable,
- arbitrary amounts of mass may move from one site to another,
- the dynamics comprises discrete (or parallel) time steps.

The class of models considered are defined on a one-dimensional lattice of L sites with periodic boundary conditions (site $L + 1 =$ site 1); associated with each site is a continuous mass variable m_i , $i = 1, \dots, L$. The total mass is given by $M = \sum_{i=1}^L m_i$.

The dynamics is defined as follows: at each time step, at each site i , mass μ_i drawn from a distribution $\phi(\mu_i|m_i)$ ‘chips off’ the mass m_i , and moves to site $i + 1$. Thus all sites are updated in parallel and mass is transferred simultaneously between sites.

6.3.1. Steady state factorization condition. The central result of [93] is a necessary and sufficient condition for the steady state to factorize which states that the chipping function must be of the form

$$\phi(\mu|m) = \frac{v(\mu)w(m-\mu)}{[v * w](m)}. \quad (142)$$

In this case, the single-site weights become

$$f(m) = [v * w](m), \tag{143}$$

where in expressions (142) and (143) the convolution is defined as

$$[v * w](m) \equiv \int_0^m d\mu v(\mu)w(m - \mu). \tag{144}$$

Equation (142) expresses that the hopping function should be a product of a function of μ , the mass which moves, and a function of $m - \mu$, the mass which remains, divided by a normalization which is a function of m .

The proof that (142) is necessary and sufficient is presented in [93]. Here we content ourselves with checking that it is a sufficient condition.

The steady state probabilities satisfy the following condition:

$$P(\underline{m}) = \prod_{l=1}^L \int_0^\infty dm'_l \int_0^{m'_l} d\mu_l \phi(\mu_l|m'_l) \prod_{k=1}^L \delta(m_k - m'_k + \mu_k - \mu_{k-1}) P(\underline{m}'). \tag{145}$$

The rhs of this equation is to be understood as integrating over all possible configurations before the update: site l contained mass $m'_l = m_l + \mu_l - \mu_{l-1}$ and mass μ_l moved to site $l + 1$ at the update. Assuming a factorized steady state, equation (145) reduces to

$$\prod_{l=1}^L f(m_l) = \prod_{l=1}^L \left[\int_0^{m_{l+1}} d\mu_l \right] \prod_k \phi(\mu_k|m_k + \mu_k - \mu_{k-1}) f(m_k + \mu_k - \mu_{k-1}). \tag{146}$$

Inserting (142), (143) the rhs becomes

$$\prod_{l=1}^L \left[\int_0^{m_{l+1}} d\mu_l \right] \prod_k v(\mu_k)w(m_k - \mu_{k-1}) = \prod_{l=1}^L [v * w](m_l), \tag{147}$$

where we relabelled $k = l + 1$. Thus (145) is satisfied when (142), (143) hold.

6.3.2. Continuous time limit. The continuous time case is a limit of the discrete-time case when we let the time step be dt and set $v(\mu) = \delta(\mu) + x(\mu) dt$ then take the limit $dt \rightarrow 0$. Condition (142) reduces to requiring hopping rates $\gamma(\mu|m)$ of the form

$$\gamma(\mu|m) = \frac{x(\mu)w(m - \mu)}{w(m)} \tag{148}$$

in which case $f(m) = w(m)$.

It is useful to check explicitly that this works in a way similar to section 2.2. The steady state probabilities satisfy the following condition which generalizes (5)

$$0 = \sum_{l=1}^L \left[\int_0^{m_l} d\mu_{l-1} \gamma(\mu_{l-1}|\mu_{l-1} + m_{l-1}) P(\{\dots, m_{l-1} + \mu_{l-1}, m_l - \mu_{l-1}, \dots\}) - \int_0^{m_l} d\mu_l \gamma(\mu_l|m_l) P(\{n_l\}) \right]. \tag{149}$$

Assuming a factorized steady state and equating terms l in the sum, yields

$$\begin{aligned} & \int_0^{m_l} d\mu_{l-1} \gamma(\mu_{l-1}|\mu_{l-1} + m_{l-1}) f(m_{l-1} + \mu_{l-1}) f(m_l - \mu_{l-1}) \\ &= \int_0^{m_l} d\mu_l \gamma(\mu_l|m_l) f(m_{l-1}) f(m_l). \end{aligned} \tag{150}$$

Inserting (148) and $f(m) = w(m)$ yields the required identity.

The case of discrete masses is easily obtained by restricting $\mu, m - \mu$ to be positive integers in (142) and defining the convolution as

$$[v * w](m) \equiv \sum_{\mu=0}^m v(\mu)w(m - \mu). \quad (151)$$

6.3.3. Heterogeneous case. We also note that the condition (144) can be generalized to the heterogeneous case where $\phi_l(\mu|m)$ depends on the site l . A necessary and sufficient condition for a factorized steady state is that

$$\phi_l(\mu|m) = \frac{v(\mu)w_l(m - \mu)}{[v * w_l](m)}, \quad (152)$$

where v and w_l are arbitrary functions but v must be the same for each site. The single-site weights are given by

$$f_l(m) = [v * w_l](m). \quad (153)$$

The continuous time limit implies that the site-dependent hopping rates should be of the form

$$\gamma(\mu|m) = \frac{x(\mu)w_l(m - \mu)}{w_l(m)}, \quad (154)$$

in which case $f_l(m) = w_l(m)$.

6.3.4. Test for factorization. Although (142) is appealingly simple, it does not tell us directly if a given hopping function ϕ yields a factorized steady state. Happily, a direct test is easily constructed [94]. We require that

$$\left. \frac{\partial}{\partial \mu} \right|_{\sigma} \left. \frac{\partial}{\partial \sigma} \right|_{\mu} \ln \phi(\mu|\mu + \sigma) = h(\mu + \sigma), \quad (155)$$

i.e., the result of differentiating $\ln \phi$ with respect to μ , the mass that moves, and σ , the mass that remains, is a function of $m = \sigma + \mu$ alone. Then from (142) we would have $h(m) = -d^2 \ln f(m)/dm^2$ and integrating twice yields

$$f(m) = \exp \left[- \int^m dm' \int^{m'} dm'' h(m'') \right]. \quad (156)$$

In the same way, we can construct a test for the integer mass case: we require

$$\frac{\phi(\mu + 1|n + 2)\phi(\mu|n)}{\phi(\mu + 1|n + 1)\phi(\mu|n - 1)} = R(n), \quad (157)$$

i.e., the cross ratios on the lhs (defined when *all* of the ϕ are positive) are functions of n alone. Then from (142) we would have

$$R(n) = \frac{f(n + 1)^2}{f(n)f(n + 2)}, \quad (158)$$

which implies the recursion

$$\frac{f(n + 2)}{f(n + 1)} = \frac{1}{R(n)} \frac{f(n + 1)}{f(n)}. \quad (159)$$

Iterating (159) twice yields

$$f(n) = f(0) \left(\frac{f(1)}{f(0)} \right)^{n-2} \prod_{j=0}^{n-2} \left[\prod_{k=0}^j \frac{1}{R(k)} \right] \quad \text{for } n \geq 2. \quad (160)$$

6.3.5. *ZRP with parallel dynamics.* As an illustration of the use of these tests and the construction of factorized steady let us first consider the ZRP with parallel dynamics. At each time step a particle hops forward from a site with n particles with probability $u(n)$. Then $\phi(0|n) = 1 - u(n)$, $\phi(1|n) = u(n)$, $\phi(k|n) = 0$ for $k > 1$.

The cross ratio in (157) is only defined for $\mu = 0$, therefore

$$R(n) = \frac{u(n+2)u(n)}{u(n+1)u(n-1)}, \tag{161}$$

is automatically a function of n alone and factorization is guaranteed. Expression (160) becomes

$$f(n) = f(0) \left(\frac{f(1)}{f(0)} \right)^n \frac{u(1)^n}{1-u(n)} \prod_{j=1}^n \frac{1-u(j)}{u(j)}. \tag{162}$$

This expression was first derived in the exclusion process context [95] by a more complicated approach. Also in that work expressions for $f(m)$ were presented for the case of ordered sequential dynamics where one updates either in the forward sequence $l = 1, \dots, L$ or the backward sequence $l = L, \dots, 1$.

6.3.6. *ZRP with transfer of more than one mass unit.* As a more complicated example we consider a generalized ZRP where mass ‘chunks’ of size one or two can chip off at each time step with probabilities $u_1(n)$ and $u_2(n)$ respectively. In this case we have two cross ratios in (157) defined for $\mu = 0, 1$. Demanding that these be equal implies that

$$R(n) = \frac{u_1(n+2)[1-u_1(n)-u_2(n)]}{u_1(n+1)[1-u_1(n+1)-u_2(n+1)]} \tag{163}$$

$$= \frac{u_2(n+2)u_1(n)}{u_2(n+1)u_1(n+1)} \quad \text{for } n \geq 1, \tag{164}$$

which reduces to

$$\frac{u_2(n+1)(1-u_1(n)-u_2(n))}{u_1(n+1)u_1(n)} = A \quad \text{for } n \geq 1, \tag{165}$$

where A is a positive constant independent of n . This recursion fixes $u_2(n)$ in terms of A , $u_2(1)$ and $u_1(n)$. Then we can use (160) and (163) to deduce the single-site weights

$$f(n) = f(0) \left(\frac{f(1)}{f(0)} \right)^n \frac{u_1(1)^n}{1-u_1(n)-u_2(n)} \prod_{j=1}^n \frac{1-u_1(j)-u_2(j)}{u_1(j)}. \tag{166}$$

6.3.7. *Condensation.* We now analyse the condensation phenomena arising from $f(n)$ of the form (143). For brevity, we restrict our attention to the homogeneous case. The grand canonical partition function becomes

$$\mathcal{Z}(z) = \left[\int_0^\infty dm z^m [v * w](m) \right]^L = [V(z)W(z)]^L, \tag{167}$$

where

$$V(z) = \int_0^\infty d\mu z^\mu v(\mu) \quad W(z) = \int_0^\infty d\sigma z^\sigma w(\sigma). \tag{168}$$

As in (32), (35) the mass density ρ determines the fugacity z through

$$\rho = \frac{zV'(z)}{V(z)} + \frac{zW'(z)}{W(z)}. \tag{169}$$

As before the condensation mechanism is as follows: since (169) is an increasing function of z one can always find a finite solution of equation (169) provided the rhs diverges at the radius of convergence of V or W . On the other hand, a condensation transition can occur if $zW'(z)/W(z)$ and $zV'(z)/V(z)$ have finite limits as z tends to the lower of the radii of convergence of $V(z)$, $W(z)$.

Note that the condensation mechanism may be driven by either or both of $v(\mu)$, $w(\sigma)$ according to which of $V(z)$, $W(z)$ has the smaller radius of convergence. The condensation in the ZRP corresponds to condensation driven by $w(\sigma)$ since in the ZRP the amount of mass that can chip off is restricted to one unit. Condensation driven by $v(\mu)$ would require a slowly decaying $v(\mu)$ which would correspond to chipping off large fractions of mass. One would expect that under these conditions the condensate is mobile since events where a large fraction of mass is chipped off will move the condensate around the lattice.

Thus in addition to the two types of condensation seen in the ZRP, i.e., condensation of particles onto the site with the slowest hopping rate in a heterogeneous system and condensation due to slowly decaying hop rates on a homogeneous system, we identify a third type of condensation due to slowly decaying size distribution of the chunks of mass transferred.

6.4. ZRP with non-conservation

So far we have considered the ZRP and generalizations where the particle number, or mass, and site number are conserved. In this section, we explore some cases where these quantities are allowed to fluctuate yet the factorization properties are preserved. These dynamics allow one to generate different ensembles from the canonical ensemble. As we shall see, this admits a variety of new transitions.

We restrict ourselves to periodic boundary conditions although we note that an open boundary ZRP has recently been considered in [96].

6.4.1. Fluctuating particle number and generation of grand canonical ensemble. Let us consider dynamics where in addition to the hopping rates $u(n)$, a particle is created at site l with rate $c(n_l)$ and a particle is annihilated with rate $a(n_l)$. In order for the steady state to remain factorized and for the particle number to fluctuate, the dynamics for the creation and annihilation of particles must obey detailed balance with respect to the steady state

$$P(\{n_1 \dots n_L\}) = \frac{\prod_l [z^{n_l} f(n_l)]}{\mathcal{Z}} \quad (170)$$

where $f(n) = f(n-1)/u(n)$ satisfies the usual factorization condition (8). Detailed balance with respect to particle non-conservation at all sites implies $a(n)f(n)z^n = c(n-1)f(n-1)z^{n-1}$ so that

$$a(n)z = u(n)c(n-1). \quad (171)$$

A convenient solution of this equation is $a(n) = u(n)$ and $c(n) = z$. Thus, with this choice of rates one generates in the steady state the grand canonical ensemble of section 4.1.

If $c(n)/a(n) > 1$ as $n \rightarrow \infty$, the particle number will diverge and a steady state will no longer be attained. In section 4.1 a cut-off on particle number, which implies $f(m) = 0$ for $m > N$, was introduced to circumvent this problem. In order to generate the cut-off dynamically one sets $f(N+1) = f(N)/\kappa$ where $\kappa \rightarrow \infty$. Then one can take $u(N+1) = a(N+1) = \kappa \rightarrow \infty$.

Finally, in section 4.1 z was taken as dependent on the total particle number $z(N)$. This can be generated dynamically by letting the creation rate be $c = z(N)$. Note that such a

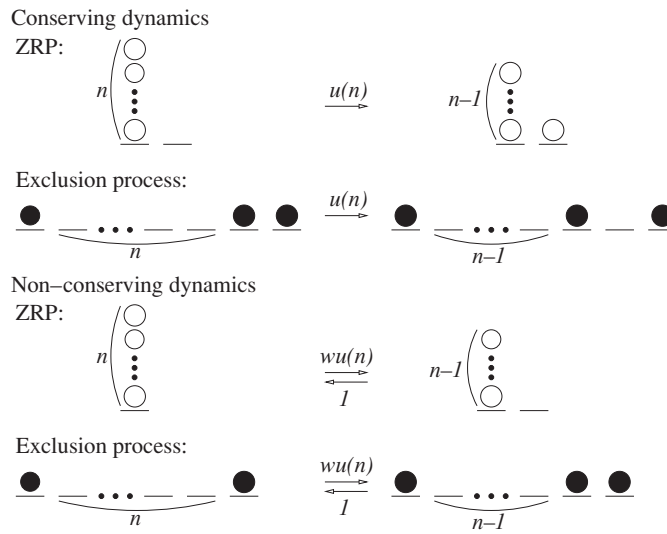


Figure 9. Illustration of the dynamics which allows the site number and the particle number to fluctuate with $L+N$ fixed. The conserving dynamics, as shown at the top of the figure, are the usual ZRP dynamics where particles hop to the neighbouring site with rate $u(n)$. The corresponding exclusion process dynamics are illustrated beneath. The lower part of the figure illustrates the non-conserving dynamics: simultaneously a particle is annihilated and an adjacent vacant site created with rate $wu(n)$ and the reverse process occurs with rate 1. The corresponding exclusion process dynamics are shown at the bottom of the figure.

dependence of local particle creation rate on the global number of particles in the system implies non-local dynamics.

6.4.2. Fluctuating site and particle number. We now consider a microscopic dynamics which allows the number of sites L and the number of particles N to fluctuate under the constraint that the total number of sites plus particles remains fixed: $L + N = K$. The dynamics is illustrated in figure 9 along with the mapping to an exclusion process.

It is easy to check that this choice of creation and annihilation dynamics satisfies detailed balance with respect to the steady state

$$P(\{n_1 \dots n_L\}) \propto \prod_l [wz^{n_l+1} f(n_l)] \tag{172}$$

where, as usual, $f(n) = f(n-1)/u(n)$. That is, the creation of new sites balances annihilation of sites through

$$wu(n)[wz^{n+1} f(n)] = [wz^n f(n-1)][wzf(0)]. \tag{173}$$

An interesting phase transition occurs under these dynamics which is most easily studied in a grand canonical ensemble. We define the grand canonical partition function as

$$\mathcal{Z}(z) \equiv \sum_{L=1}^{\infty} (wz)^L \prod_{l=1}^L \sum_{n_l=0}^{\infty} z^{n_l} f(n_l) \tag{174}$$

$$= \frac{wzF(z)}{1 - wzF(z)} \tag{175}$$

where $F(z)$ is defined as in (34). The value of z is fixed through the condition

$$K = z \frac{\partial \ln \mathcal{Z}}{\partial z} = \langle L \rangle [\langle n \rangle + 1] \quad (176)$$

where

$$\langle L \rangle = w \frac{\partial \ln \mathcal{Z}}{\partial w} = \frac{1}{1 - wzF(z)} \quad (177)$$

$$\langle n \rangle = \frac{zF'(z)}{F(z)}. \quad (178)$$

$\langle L \rangle$ is the average number of sites and $\langle n \rangle$ is the average number of particles per site. When K is large, (176) is satisfied when either $\langle L \rangle$ or $\langle n \rangle$ is large.

Let us take as example the solvable case of section 4.4.1: $u(n) = 1 + b/n$ and for simplicity we consider only b non-integer.

If $b < 1$, $F(1)$ diverges and we can clearly find $z < 1$ such that $1 - wzF(z) = \mathcal{O}(1/K)$. Thus for $b < 1$ we have from (177), (178) that $\langle L \rangle = \mathcal{O}(K)$ and $\langle n \rangle = \mathcal{O}(1)$. This implies a number of sites of order K each with a finite number of particles.

For $b > 1$ we have the explicit expression $F(1) = b/(b-1)$. Thus if $w > w_c = (b-1)/b$ we can again choose z to give $\langle L \rangle = \mathcal{O}(K)$ and $\langle n \rangle = \mathcal{O}(1)$. For $w < w_c$, on the other hand, we have different behaviour according to whether $b < 2$ or $b > 2$.

For $b < 2$, $F'(1)$ diverges. Therefore for $w < w_c$ we can choose $z \nearrow 1$ such that $\langle n \rangle = \mathcal{O}(K)$. In this case $\langle L \rangle = \mathcal{O}(1)$, thus we have a finite number of sites each with a number of particles of order K .

For $b > 2$, $F'(1) = b/(b-1)(b-2)$, and for $w < w_c$ we cannot satisfy (176) as $z \nearrow 1$. Moreover as $z \nearrow 1$, both $\langle L \rangle$ and $\langle n \rangle$ are finite. Therefore we must have a condensation transition wherein a condensate site emerges which contains fraction one of the particles in the large K limit.

The mechanism for these transitions was first studied in a simple lattice model for DNA denaturation [64]. Within the mapping to an exclusion process the constraint that K is constant corresponds to a fixed number of lattice sites with particle non-conservation. Thus the transition is from a state with a finite density of particles to a zero density state which comprises a finite number of particles with large gaps between for $1 < b < 2$, or the condensed state with one large gap and a small jam of closely spaced particles for $b > 2$.

It has been shown how the transition may occur in an exclusion process with local dynamics by using two species of particles to generate *effective* hop rates such that $b = 3/2$ and $f(n) \sim 1/n^{3/2}$, in the spirit of section 3.3.2 [97].

6.4.3. Fluctuating site number only. One can also consider the case where only the site number L is allowed to fluctuate [98]. For dynamics in which, in addition to the usual ZRP hop rates, vacant sites can be created with rate w and annihilated with rate 1, the steady state

$$P_L(\{n_1 \dots n_L\}) \propto w^L \prod_{l=1}^L f(n_l), \quad (179)$$

satisfies detailed balance with respect to the creation and annihilation dynamics. The L subscript here is intended to emphasize that this is the probability for a configuration of N particles on exactly L sites.

This model undergoes a transition which can be analysed in a way following just the same steps as the previous case, so we neglect the details and quote only the results. We consider hop rates $u(n) = 1 + b/n$, and we consider the three quantities N , $\langle L \rangle$ and $\langle n \rangle$.

There exists some critical value of w , w_c , such that if $w > w_c$ then we can work in the grand canonical ensemble and find a fugacity such that $\langle L \rangle = \mathcal{O}(N)$ and $\langle n \rangle = \mathcal{O}(1)$. However for $w < w_c$ two cases ensue. If $b < 2$, one can find a fugacity such that $\langle n \rangle = \mathcal{O}(N)$ and $\langle L \rangle = \mathcal{O}(1)$. On the other hand, if $b > 2$ one finds that both $\langle L \rangle = \mathcal{O}(1)$ and $\langle n \rangle = \mathcal{O}(1)$. Therefore in the limit $N \rightarrow \infty$, a single site contains a fraction equal to one of the N particles in the system.

7. Summary and open questions

In this work, we have reviewed the properties of the zero-range process and a variety of generalizations and related models. We have focused on the property of a factorized steady state which is exhibited in many of these models: the ZRP always factorizes, whereas generalizations such as the two-species model (section 6.1) and Misanthrope process (section 6.2), where the hop rate also depends on the target site, only factorize when the hop rates satisfy certain specific conditions (97), (129). The generalization of section 6.3, to continuous mass, discrete time and arbitrary amount of mass transferred, yields a condition for factorization with an appealingly simple general form. The factorization property allows an exact analysis of steady state behaviour, in particular condensation. It should be noted that condensation is not restricted to models with factorized steady states, rather factorization has offered so far the only opportunity to study condensation exactly. We remark that factorization has also been used as a mean-field-type approximation to study condensation in systems where the steady state is not known exactly.

More generally, it would be of interest to study the structure of non-factorized steady states and how condensation arises. For example, one generalization of a factorized steady state is to a matrix product state [53].

Returning to the case of factorized steady state, there have been a number of developments in our understanding of condensation. Here we have reviewed the distinct mechanisms of condensation in homogeneous systems (sections 4 and 6.3.7), heterogeneous systems (section 5), including the case of a single defect site (section 5.2), non-conserving systems (section 6.4) and systems with more than one species of particles (section 6.1). We have also seen an example where the condensate is mobile rather than fixed (section 6.3.7). In particular, the analysis of condensation within the canonical ensemble (section 4.2) should allow a deeper understanding of the nature of condensates.

We have also reviewed the dynamics of condensation. The coarsening dynamics is now typically understood at a mean-field level or using heuristic random walk arguments. These arguments have been applied to condensation in homogeneous (section 4.5), heterogeneous (section 5.3) and two-species systems (section 6.1.3). An open question is whether one can obtain scaling functions for dynamics beyond the mean-field level. This would enable one to address questions of universality. We also note that for certain parameters (for which condensation does not occur) the Bethe ansatz has been used to calculate exactly dynamical properties of the ZRP [99].

Finally, we are aware that this review is of a theoretical nature and we have not emphasized or speculated upon experimental realizations. In section 3.1 we discussed clustering in shaken granular gases, which furnishes a pleasing experimental example of condensation in interacting particle systems. Another realization of the ZRP noted in [100] concerns the exchange of monomers between protein filaments. It remains an important challenge to establish further experimental instances of interacting particle systems and the associated phenomena reviewed here.

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